Refined GW/Kronecker correspondence

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January 18, 2013

Abstract

Gromov-Witten invariants of weighted projective planes and Euler characteristics of moduli spaces of representations of bipartite quivers are related via the tropical vertex, a group of formal automorphisms of a torus. On the Gromov-Witten side, this uses the work of Gross, Pandharipande and Siebert. The quiver moduli side features quiver wall-crossing formulas, functional equations for Euler characteristics, and localization techniques. We derive several explicit formulas for Gromov-Witten invariants.

1 Introduction

It was noted in [3], [11], and explained in detail in [2], that there is a numerical correspondence (named GW/Kronecker correspondence in [19]) between two seemingly unrelated geometries :

The first geometry is the Gromov-Witten theory of weighted projective planes, with Gromov-Witten invariants counting rational maps to open parts of weighted projective planes intersecting the toric divisors in prescribed points with prescribed multiplicities (see [3, Section 0]). The second geometry is the theory of moduli spaces of quiver representations (see for example [13]), and the relevant numerical information is just the Euler characteristic of these moduli.

The relation between these geometries is realized in the tropical vertex, a group of formal automorphisms of a torus which first appeared in [7] and plays a prominent role in describing the wall-crossing behaviour of Donaldson-Thomas invariants in [6]. Namely, generating series of Gromov-Witten invariants, respectively of Euler characteristics of quiver moduli, appear in a natural factorization of a commutator in the tropical vertex.

On the Gromov-Witten side, this result of [3] requires a passage from the above Gromov-Witten invariants via degeneration formulas and multiple cover calculations in Gromov-Witten theory, and holomorphic and tropical curve counts, to scattering diagrams describing factorizations in the tropical vertex.

On the quiver moduli side, the wall-crossing formula of [11] is derived via counting points of quiver moduli over finite fields, realizing the Harder-Narasimhan recursion as

an identity in Hall algebras of quivers, and translating this identity to a group of Poisson automorphisms using framed versions of quiver moduli. The tropical vertex appears in the special situation of moduli spaces for generalized Kronecker quivers.

In this paper, we refine and generalize the GW/Kronecker correspondence, proving that the Euler characteristics of appropriate quiver moduli determine, and are determined by, the above Gromov-Witten invariants. In several cases, this allows us to give explicit formulas for the Gromov-Witten invariants, confirming in particular the conjecture of [2, Section 1.4] on Gromov-Witten invariants of the projective plane, and some cases of the integrality conjecture [3, Conjecture 6.2].

To achieve this, we first apply the wall-crossing formula of [11] to complete bipartite quivers; similarly to [2], this gives a correspondence between Gromov-Witten invariants and Euler characteristics of framed moduli spaces of representations of these quivers. We then apply the formalism of [12], relating Euler characteristics of framed and unframed quiver moduli by systems of functional equations for their generating series, resulting in a correspondence between Gromov-Witten invariants and Euler characteristics of unframed moduli spaces of representations of bipartite quivers. Explicit formulas are obtained via localization theory for quiver moduli as developed in [20]; several of the main results of this paper are generalized to bipartite quivers.

In sections 2 and 3, we recall the definition of the tropical vertex and the basic factorization problem (1) which is solved by the two different above geometries, as well as the relevant Gromov-Witten theory. We follow the notation of [3] very closely to allow the reader a direct comparison of our results with [2, 3].

All notions and results from the theory of representations of quivers and their moduli spaces (in particular, the definition and basic geometric properties of the unframed and framed moduli spaces, the wall-crossing formula of [11] and the functional equations of [12]), which are necessary for the derivation of the refined GW/Kronecker correspondence, are reviewed in Section 4. In sections 5 and 6, we specialize these methods to complete bipartite quivers, and obtain a first version of the refined GW/Kronecker correspondence in Theorem 6.1. The second version of the correspondence, Theorem 7.1 in Section 7, is obtained by specializing the functional equations of [12] to bipartite quivers.

Although this second version shows that Gromov-Witten invariants and Euler characteristics of quiver moduli determine each other, the precise mechanism for this determination is (in general) hidden under an infinite system of coupled functional equations, reminiscent of Q-system type equations [8] arising from the Bethe ansatz equations of solvable lattice models (although no potential lattice model corresponding to the equations of Theorem 7.1 is known). To extract more specific information from Theorem 7.1, special cases where the Euler characteristics of quiver moduli can be computed (usually by localization techniques) are considered in the following sections:

A trivial first order analysis of the functional equations yields a much simpler correspondence in the so-called coprime case in Section 9: on the Gromov-Witten side, the numbers of intersections with two of the toric divisors of the rational curves to be counted are assumed to be coprime; on the quiver moduli side, coprimality of dimension types of representations results in compact moduli. The correspondence then just states that the two invariants are equal (Corollary 9.1). For one particular slope (the ratio of the total intersection multiplicities with two toric divisors), localization methods yield an explicit formula for non-trivial counting invariants (Theorem 9.4); a conceptual reason for this particular case to be computable (and for being almost the only such example)

is still unknown.

In Section 10, we consider those (few) cases where the complete bipartite quiver is of (extended) Dynkin type; the known classification of all representations in these cases allows us to give a complete description of the factorization (1) in the tropical vertex, thereby determining all Gromov-Witten invariants counting curves intersecting the toric divisors in "few" points.

It was already noted in [2, 3] via computer experiments that a closed formula, conjectured in [2, Section 1.4], can be expected for the Gromov-Witten invariants of the (unweighted) projective plane. We prove this conjecture in Section 11 using a vanishing result, Corollary 16.4, for certain Euler characteristics (although the corresponding quiver moduli are highly non-trivial), which simplifies the functional equations to a finite set of algebraic functional equations in Theorem 11.1. A more detailed closed formula can be expected to follow from an appropriate application of multivariate Lagrange inversion.

To address integrality properties of the Gromov-Witten invariants and their associated BPS state counts (see [3, Conjecture 6.2]), we consider specialized variables in the tropical vertex, which on the Gromov-Witten side means that only the total numbers of intersections of curves with the toric divisors are recorded. We show in Section 8 that the system of functional equations reduces to a single such equation (Theorem 8.1) of the type which was already studied in [12] in connection with the relative integrality of Donaldson-Thomas invariants. The methods of [12], together with a subtle divisibility property for the Euler characteristic of quiver moduli (Theorem 12.1) obtained again by a localization argument, yield integrality in the so-called balanced case (see Section 12).

We briefly consider more general commutator formulas in the tropical vertex [3, Theorem 5.6] in Section 13. This requires consideration of moduli spaces for bipartite quivers with level structure. However, the analogue of the GW/Kronecker correspondence, Theorem 13.1, is weaker in this generality; the reason for this is unclear at the moment.

Sections 14 to 17 develop the localization techniques which, as indicated above, form the main technical tool for derivation of explicit formulas for Gromov-Witten invariants. After reviewing general concepts in Section 14, the two explicit formulas for Euler characteristics leading to Theorem 9.4 and Corollary 12.2 are developed in Section 15. Several situations (in the generality of levelled bipartite quivers) in which Euler characteristics of quiver moduli vanish are derived in Section 16. Finally, Section 17 details some further classes of examples.

In conclusion, the refined GW/Kronecker correspondence, together with localization techniques, allows to obtain several nontrivial formulas for Gromov-Witten invariants. These results hopefully serve as a starting point for investigation of a direct geometric relation between the two geometries in question.

2 The tropical vertex

We review the definition of the tropical vertex following [3, Section 0].

We fix nonnegative integers $l_1, l_2 \geq 1$ and define R as the formal power series ring $R = \mathbb{Q}[[s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2}]]$, with maximal ideal $\mathfrak{m}$. Let B be the R-algebra

$$B = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}][[s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2}]] = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]\widehat{\otimes} R$$

(a suitable completion of the tensor product). We consider R-linear automorphisms of B (more precisely, we only consider automorphisms respecting the symplectic form $\frac{dx}{x} \wedge \frac{dy}{y}$):

For $(a, b) \in \mathbb{Z}^2$ and a series $f \in 1 + x^a y^b \mathbb{Q}[x^a y^b]\widehat{\otimes}\mathfrak{m}$, define the automorphism

$$T_{(a,b),f} : \begin{cases} x & \mapsto & xf^{-b} \\ y & \mapsto & yf^a. \end{cases}$$

Definition 2.1 *The tropical vertex group $H \subset \operatorname{Aut}_R(B)$ is defined as the completion with respect to $\mathfrak{m}$ of the subgroup of $\operatorname{Aut}_R(B)$ generated by all elements $T_{(a,b),f}$ as above.*

By [7] (see also [3, Theorem 1.3]), there exists a unique factorization in H into an infinite ordered product

$$T_{(1,0),\prod_k (1+s_k x)} T_{(0,1),\prod_l (1+t_l y)} = \prod_{b/a \text{ decreasing}} T_{(a,b),f_{(a,b)}}, \tag{1}$$

the product ranging over all coprime pairs $(a, b) \in \mathbb{N}^2$. The main problem addressed in [3] is to describe the series $f_{(a,b)}$ appearing in this factorization.

As an example, we list the cases in which the factorization (1) actually involves only finitely many terms; these are exactly the cases where $l_1 l_2 \leq 3$. Without loss of generality, we can assume $l_1 \leq l_2$; we borrow results from Section 10:

If $l_1 = 1$, $l_2 = 1$, we have $T_{(1,0),1+s_1 x} T_{(0,1),1+t_1 y} =$

$$= T_{(0,1),1+t_1 y} T_{(1,1),1+s_1 t_1 xy} T_{(1,0),1+s_1 x}.$$

If $l_1 = 1$, $l_2 = 2$, we have $T_{(1,0),1+s_1 x} T_{(0,1),(1+t_1 y)(1+t_2 y)} =$

$$= T_{(0,1),(1+t_1 y)(1+t_2 y)} T_{(1,2),1+s_1 t_1 t_2 xy^2} T_{(1,1),(1+s_1 t_1 xy)(1+s_1 t_2 xy)} T_{(1,0),1+s_1 x}.$$

If $l_1 = 1$, $l_2 = 3$, we have $T_{(1,0),1+s_1 x} T_{(0,1),(1+t_1 y)(1+t_2 y)(1+t_3 y)} =$

$$= T_{(0,1),(1+t_1 y)(1+t_2 y)(1+t_3 y)} T_{(1,3),1+s_1 t_1 t_2 t_3 xy^3} T_{(1,2),(1+s_1 t_1 t_2 xy^2)(1+s_1 t_1 t_3 xy^2)(1+s_1 t_2 t_3 xy^2)}$$

$$T_{(2,3),1+s_1^2 t_1 t_2 t_3 x^2 y^3} T_{(1,1),(1+s_1 t_1 xy)(1+s_1 t_2 xy)(1+s_1 t_3 xy)} T_{(1,0),1+s_1 x}.$$

There are two more cases where the factorization (1), although involving infinitely many nontrivial factors, can be described completely; these are the cases $l_1 l_2 = 4$ to be discussed in Section 10.

3 Gromov-Witten invariants

The main result of [3] is the description of the series $f_{(a,b)}$ appearing in the factorization (1) in terms of Gromov-Witten theory of certain toric surfaces:

We follow the notation of [3, Section 0.4]. Let $\Sigma \subset \mathbb{Z}^2$ be the fan with rays generated by $-(1,0), -(0,1), (a,b)$. Let $X_{a,b}$ be the toric surface over $\mathbb{C}$ associated to Σ (which

is isomorphic to the weighted projective plane $(\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$ for the action $t(x, y, z) = (t^a x, t^b y, tz))$ with corresponding toric divisors $D_1, D_2, D_{\mathrm{out}}$. Let $X_{a,b}^o \subset X_{a,b}$ be the open surface obtained by removing the toric fixed points, and let $D_1^o, D_2^o, D_{\mathrm{out}}^o$ be the restrictions of the toric divisors to $X_{a,b}^o$.

We consider a pair $(\mathbf{P}_1, \mathbf{P}_2)$ of ordered partitions (that is, partitions with nonnegative parts which are allowed to be 0, and whose order is kept track of), written

$$\mathbf{P}_1 = p_{1,1} + \ldots + p_{1,l_1}, \quad \mathbf{P}_2 = p_{2,1} + \ldots + p_{2,l_2},$$

such that $|\mathbf{P}_1| = \sum_{l=1}^{l_1} p_{1,l} = ka$ and $|\mathbf{P}_2| = \sum_{l=1}^{l_2} p_{2,l} = kb$ for some $k \geq 1$. Let $\nu : X_{a,b}[(\mathbf{P}_1, \mathbf{P}_2)] \to X_{a,b}$ be the blow-up of $X_{a,b}$ along l_1 (resp. l_2) points of D_1^o (resp. D_2^o), and define $X_{a,b}^o[(\mathbf{P}_1, \mathbf{P}_2)] = \nu^{-1}(X_{a,b}^o)$. Let $\beta_k \in H^2(X_{a,b}, \mathbb{Z})$ be the unique cohomology class with intersection numbers

$$\beta_1 \cdot D_1 = ka, \ \beta_2 \cdot D_2 = kb, \ \beta_k \cdot D_{\mathrm{out}} = k.$$

Define a cohomology class $\beta_k[(\mathbf{P}_1, \mathbf{P}_2)] \in H^2(X_{a,b}[(\mathbf{P}_1, \mathbf{P}_2)], \mathbb{Z})$ by

$$\beta_k[(\mathbf{P}_1, \mathbf{P}_2)] = \nu^*(\beta_k) - \sum_{k=1}^{l_1} p_{1,k}[E_{1,k}] - \sum_{l=1}^{l_2} p_{2,l}[E_{2,l}],$$

where $E_{i,k}$ for $k = 1, \ldots, l_i$ denotes the k-th exceptional divisor over D_i^o for $i = 1, 2$. The moduli space $\overline{\mathfrak{M}}(X_{a,b}^o[(\mathbf{P}_1, \mathbf{P}_2)]/D_{\mathrm{out}}^o)$ of genus 0 maps to $X_{a,b}^o[(\mathbf{P}_1, \mathbf{P}_2)]$ in class $\beta_k[(\mathbf{P}_1, \mathbf{P}_2)]$ with full contact order k at an unspecified point of D_{out}^o is proper and of virtual dimension 0, thus a corresponding Gromov-Witten invariant $N_{a,b}[(\mathbf{P}_1, \mathbf{P}_2)] \in \mathbb{Q}$ is well-defined (see [3, Section 5.2]).

Heuristically, $N_{a,b}[(\mathbf{P}_1, \mathbf{P}_2)]$ may be viewed as the "number" of rational curves in $X_{a,b}$ intersecting the distinct fixed l_i points of D_i^o with multiplicities given by the $p_{i,l}$ for $i = 1, 2$, and being tangent to D_{out}^o of order k. But note that this counting is only straightforward in very particular cases; in general, complicated contributions from degenerations and multiple covers have to be accounted for.

Example 3.1

We have $N_{(1,3)}[(1, 1+1+1)] = 1$, since there is (up to reparametrization) a unique curve

$$(u : v) \mapsto (u : -\frac{y_1}{x_1 x_2 x_3}(u - x_1 v)(u - x_2 v)(u - x_3 v) : v)$$

from $\mathbf{P}^1$ to $X_{1,3}$ intersecting D_1^o transversally in three points x_1, x_2, x_3, intersecting D_2^o transversally in one point y_1, and intersecting D_{out}^o transversally in one point.

We also have $N_{(2,3)}[2, 1+1+1] = 1$, but the counting procedure is more indirect: choosing signs $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$, there are four curves

$$(u : v) \mapsto (u^2 : -\frac{y_1}{x_1 x_2 x_3}(u - \varepsilon_1 x_1 v)(u - \varepsilon_2 x_2 v)(u - \varepsilon_3 x_3 v) : v)$$

from $\mathbf{P}^1$ to $X_{2,3}$ intersecting D_1^o transversally in three points x_1^2, x_2^2, x_3^2, intersecting D_2^o with multiplicity two in one point y_1, and intersecting D_{out}^o transversally in one point. By the degeneration formula [3, Proposition 5.3], these have to be weighted by the factor $-\frac{1}{2}$ in the computation of $N_{(2,3)}[2, 1+1+1]$, and six curves intersecting D_2^o transversally in two distinct points have to be added with a weight factor of $\frac{1}{2}$.

We have $N_{(1,1)}[(1+1, 1+1)] = 2$, since there are two curves

$$(u : v) \mapsto (u(u-v) : (u-2v)(u-4v) : v^2),$$

$$(u : v) \mapsto (u(u - \frac{5}{\sqrt{3}}v) : -(u - 2\sqrt{3}v)(u + \frac{4}{\sqrt{3}}v) : v^2)$$

intersecting D_1^o transversally in the two points 2, 12, intersecting D_2^o transversally in the two points 3, 8, and being tangent of order 2 to D_{out}^o.

See also [3, Section 6.4] for some more examples.

The main result of [3] is

Theorem 3.2 *[3, Theorem 5.4] For all coprime (a,b), we have*

$$\log f_{(a,b)} = \sum_{k \geq 1} \sum_{\substack{|\mathbf{P}_1| = ka, \\ |\mathbf{P}_2| = kb}} k N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] s^{\mathbf{P}_1} t^{\mathbf{P}_2} (x^a y^b)^k.$$

4 Recollections on quiver moduli, wall-crossing and functional equations

Let Q be a finite quiver, given by a finite set Q_0 of vertices and finitely many arrows $\alpha : i \to j$. We denote by $\Lambda = \mathbb{Z}Q_0$ the free abelian group over Q_0, by $\Lambda^+ = \mathbb{N}Q_0 \subset \Lambda$ the set of dimension vectors, which will be written as $d = \sum_{i \in Q_0} d_i i \in \Lambda^+$, and by $'\Lambda^+ = \Lambda^+ \setminus \{0\}$ the set of nonzero dimension vectors. The Euler form on Λ is given by

$$\langle d, e \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha : i \to j} d_i e_j.$$

A representation V of Q of dimension vector $d \in \Lambda$ (over the field of complex numbers) consists of complex vector spaces V_i for $i \in Q_0$ of dimension d_i, and of linear maps $V_\alpha : V_i \to V_j$ for every arrow $\alpha : i \to j$ in Q.

We discuss the roots of a quiver; see [5] for more details. A dimension vector is called a root if there exists at least one indecomposable representation of this dimension and it is called Schur root if, in addition, the endomorphism ring of at least one representation is trivial. The last condition already implies that an open subset of representations has trivial endomorphism ring, see for instance [17]. A root $d \in \mathbb{N}Q_0$ is called real if we have $d \in W(Q)Q_0$, i.e. d is a reflection of a simple root, where $W(Q)$ denotes the Weyl group of the quiver. All the other roots are called imaginary. It is well-known that a root is real if and only if $\langle d, d \rangle = 1$ and imaginary if and only if $\langle d, d \rangle \leq 0$. If $\langle d, d \rangle = 0$, we also call d isotropic.

Let $(d, e) := \langle d, e \rangle + \langle e, d \rangle$ be the symmetrized Euler form. The fundamental domain $F(Q)$ of $\mathbb{N}Q_0$ is given by the dimension vectors d such that $(d, q) \leq 0$ for all $q \in Q_0$. Moreover, we have $d \in W(Q)F(Q)$ for all imaginary roots d.

Let $\Theta \in \Lambda^*$ be a functional on Λ, viewed as a stability. It induces a slope function $\mu : '\Lambda^+ \to \mathbb{Q}$ by $\mu(d) = \Theta(d)/\dim d$, where $\dim \in \Lambda^*$ is the functional $\dim d = \sum_{i \in Q_0} d_i$ (in Section 13, we will consider more general stabilities: additionally to the functional Θ, we choose a functional κ assuming positive values on $'\Lambda^+$ and define the slope function by $\mu(d) = \Theta(d)/\kappa(d)$; all results reviewed in the following are easily seen to generalize

to this context with obvious modifications). For $\mu \in \mathbb{Q}$, let $'\Lambda_\mu^+$ be the set of dimension vectors $d \in '\Lambda^+$ of slope μ, and let Λ_μ^+ be $'\Lambda_\mu^+ \cup \{0\}$, a subsemigroup of Λ^+.

The slope of a representation V of Q is defined as the slope of its dimension vector. The representation V is called Θ-(semi-)stable if the slope (weakly) decreases on proper non-zero subrepresentations.

There exists a moduli space $M_d^{\Theta-\mathrm{st}}(Q)$ for isomorphism classes of Θ-stable representations of Q of dimension vector d. If non-empty, it is a smooth irreducible variety of dimension $1 - \langle d, d \rangle$. It is projective if Q has no oriented cycles and d is Θ-coprime, that is, if $\mu(e) = \mu(d)$ for $0 \neq e \leq d$ implies $e = d$. Call a dimension vector $d \in \Lambda^+$ indivisible if $\gcd(d_i : i \in Q_0) = 1$. For generic Θ, a dimension vector d is Θ-coprime if and only if it is indivisible.

For a Θ-coprime dimension vector d, there is an explicit formula for the Poincaré polynomial of $M_d^{\Theta-\mathrm{st}}(Q)$ arizing from a resolution of a Harder-Narasimhan type recursion:

Theorem 4.1 *[9, Corollary 6.8] For Θ-coprime d, we have*

$$\sum_i \dim H^i(M_d^{\Theta-\mathrm{st}}(Q), \mathbb{Q})q^{i/2} = (q-1)\sum_{d^*}(-1)^{s-1}q^{-\sum_{k \leq l}\langle d^l, d^k \rangle}\prod_{k=1}^s \prod_{i \in Q_0}\prod_{j=1}^{d_i^k}(1 - q^{-j})^{-1},$$

where the sum ranges over all decompositions $d = d^1 + \ldots + d^s$ of d such that all d^k are non-zero, and $\mu(d^1 + \ldots + d^k) > \mu(d)$ for all $k < s$.

In sections 14, 15, 16, we will make key use of localization theory for quiver moduli as developed in [20]. The starting point for this method is the fact that the Euler characteristic $\chi(M_d^{\Theta-st}(Q))$ for a quiver containing (unoriented) cycles can be computed in terms of the Euler characteristics of various moduli spaces of representations of the universal covering $\tilde{Q}$ of Q (for precise definitions and the proof, see Section 14):

Theorem 4.2 *Let Q be a quiver with dimension vector d. Then for the Euler characteristic of the moduli space $M_d^{\Theta-\mathrm{st}}(Q)$ we have*

$$\chi(M_d^{\Theta-\mathrm{st}}(Q)) = \sum_{\tilde{d}} \chi(M_{\tilde{d}}^{\tilde{\Theta}-\mathrm{st}}(\tilde{Q})),$$

where $\tilde{d}$ ranges over all equivalence classes being compatible with d.

Let $n \in \Lambda^+$ be another dimension vector, and choose a complex vector space W_i of dimension n_i for each vertex $i \in Q_0$. There exists a moduli space $M_{d,n}^\Theta(Q)$ parametrizing equivalence classes of pairs (V, f), where V is a Θ-semistable representation of Q of dimension vector d and $f = (f_i : W_i \to V_i)_{i \in Q_0}$ is an Q_0-graded linear map such that the following holds: if $U \subset V$ is a subrepresentation containing the image of f, that is, $f_i(W_i) \subset U_i$ for all $i \in Q_0$, then $\mu(U) < \mu(V)$. Such objects are parametrized up to isomorphisms of semistable representations intertwining the Q_0-graded linear maps. This moduli space is called a smooth model in [4]. If non-empty, $M_{d,n}^\Theta(Q)$ is a smooth irreducible variety of dimension $n \cdot d - \langle d, d \rangle$, where $n \cdot d = \sum_{i \in Q_0} n_i d_i$. It contains an open subset which is a $\mathbf{P}^{n \cdot d - 1}$-fibration over $M_d^{\Theta-\mathrm{st}}(Q)$; thus it can be viewed as a (partial) compactification of a projective space fibration over $M_d^{\Theta-\mathrm{st}}(Q)$ – note that no natural smooth compactifications of $M_d^{\Theta-\mathrm{st}}(Q)$ are known in general.

We define the generating series of Euler characteristics

$$Q_\mu^{(n)}(x) = \sum_{d \in \Lambda_\mu^+} \chi(M_{d,n}^\Theta(Q)) x^d \in \mathbb{Z}[[\Lambda_\mu^+]].$$

Let $\{d, e\} = \langle d, e \rangle - \langle e, d \rangle$ be the antisymmetrization of the Euler form. We assume that Q has no oriented cycles, thus we can order the vertices as $Q_0 = \{i_1, \ldots, i_r\}$ in such a way that $k > l$ provided there exists an arrow $i_k \to i_l$.

Define a Poisson algebra $B(Q) = \mathbb{Q}[[x_i \; : \; i \in Q_0]]$ with Poisson bracket $\{x^d, x^e\} = \{d, e\} x^{d+e}$, where $x^d = \prod_{i \in Q_0} x_i^{d_i}$ denotes the natural topological basis of $B(Q)$. We consider Poisson automorphisms of $B(Q)$. For a vertex $i \in Q_0$, define $T_i \in \mathrm{Aut}(B(Q))$ by $T_i(x^d) = x^d(1 + x_i)^{\{i,d\}}$.

Theorem 4.3 *[11, Theorem 2.1] We have the following factorization in $\mathrm{Aut}(B(Q))$:*

$$T_{i_1} \circ \ldots \circ T_{i_r} = \overset{\leftarrow}{\prod_{\mu \in \mathbb{Q}}} T_\mu,$$

where

$$T_\mu(x^d) = x^d \prod_{i \in Q_0} Q_\mu^i(x)^{\{i,d\}}.$$

For an arbitrary functional $\eta \in (\mathbb{Q}Q_0)^*$, we define $Q_\mu^\eta(x) = \prod_{i \in Q_0} Q_\mu^i(x)^{\eta(i)}$, thus in particular $Q_\mu^{n \cdot}(x) = Q_\mu^{(n)}(x)$. By [11, Lemma 3.6], we have:

Lemma 4.4 *The series $Q_\mu^{\Theta - \mu \dim}$ equals 1.*

The following is the main result of [12]:

Theorem 4.5 *The series $Q_\mu^{(n)}(x)$ is given by*

$$Q_\mu^{(n)}(x) = \prod_{d \in '\Lambda_\mu^+} R^d(x)^{\chi(d)(n \cdot d)},$$

where the series $R^d(x) \in \mathbb{Z}[[\Lambda_\mu^+]]$ for $d \in '\Lambda_\mu^+$ are uniquely determined by the following system of functional equations:

For all $d \in '\Lambda_\mu^+$, we have

$$R^d(x) = (1 - x^d \prod_{e \in '\Lambda_\mu^+} R^e(x)^{-\chi(e)\langle d,e \rangle})^{-1},$$

where $\chi(d) = \chi(M_d^{\Theta - \mathrm{st}}(Q))$.

5 Complete bipartite quivers

In this section, we specialize the results of the previous section to a class of complete bipartite quivers.

Let $K = K(l_1, l_2)$ be the quiver with set of vertices $Q_0 = \{i_1, \ldots, i_{l_1}, j_1, \ldots, j_{l_2}\}$, and one arrow from each vertex j_l to each vertex i_k; thus K is a complete bipartite quiver. We thus have a natural $S_{l_1} \times S_{l_2}$-symmetry of $K(l_1, l_2)$. Dimension vectors for K can

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be written as $d(\mathbf{P}_1, \mathbf{P}_2)$ for a pair $(\mathbf{P}_1, \mathbf{P}_2)$ of ordered partitions $\mathbf{P}_i = p_{i,1} + \ldots + p_{i,l_i}$ of length l_i (for $i = 1, 2$) as in Section 3, via $d(\mathbf{P}_1, \mathbf{P}_2)_{i_k} = p_{1,k}$, $d(\mathbf{P}_1, \mathbf{P}_2)_{j_l} = p_{2,l}$. We will always assume that $|\mathbf{P}_1| = ka$, $|\mathbf{P}_2| = kb$ for a and b coprime and $k \geq 1$. The Euler form on $K(l_1, l_2)$ is given by

$$\langle d(\mathbf{P}_1, \mathbf{P}_2), d(\mathbf{P}_1', \mathbf{P}_2') \rangle = \sum_k p_{1,k} p_{1,k}' + \sum_l p_{2,l} p_{2,l}' - \sum_{k,l} p_{1,k}' p_{2,l},$$

and thus its antisymmetrization is given by

$$\{d(\mathbf{P}_1, \mathbf{P}_2), d(\mathbf{P}_1', \mathbf{P}_2')\} = \sum_{k,l} (p_{1,k} p_{2,l}' - p_{1,k}' p_{2,l}).$$

We choose the stability Θ on $K(l_1, l_2)$ given by $\Theta(j_l) = 1$, $\Theta(i_k) = 0$ (in fact, among the stabilities respecting the symmetry of the quiver, this is the only nontrivial one, see [13, Section 5.1] for a discussion in the case of the m-Kronecker quiver $K(m)$).

We describe the resulting moduli spaces:

Let V_k be $\mathbb{C}$-vector spaces of dimension $p_{1,k}$ for $k = 1, \ldots, l_1$, and let W_l be vector spaces of dimension $p_{2,l}$ for $l = 1, \ldots, l_2$, respectively. Then we consider the action of the group $G = \prod_{k=1}^{l_1} \mathrm{GL}(V_k) \times \prod_{l=1}^{l_2} \mathrm{GL}(W_l)$ on the space $\bigoplus_{k,l} \mathrm{Hom}(W_l, V_k)$ of tuples $(f_{k,l} : W_l \to V_k)_{k,l}$ of linear maps by base change. Such a tuple is called stable if for all proper non-zero tuples $(U_l \subset W_l)_l$ of subspaces, we have

$$\sum_k \dim \sum_l f_{kl}(U_l) > \frac{a}{b} \sum_l \dim U_l.$$

Theorem 5.1 *The moduli space*

$$M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2) = M_{d(\mathbf{P}_1, \mathbf{P}_2)}^{\Theta-\mathrm{st}}(K(l_1, l_2))$$

parametrizes stable tuples $(f_{k,l})_{k,l}$ up to the action of G. If non-empty, it is a smooth and irreducible variety of dimension

$$1 - \sum_i p_{1,i}^2 - \sum_j p_{2,j}^2 + \sum_{k,l} p_{1,k} p_{2,l}.$$

It is projective if $k = 1$, that is, if $|\mathbf{P}_1|$ and $|\mathbf{P}_2|$ are coprime.

We discuss the root system of the bipartite quiver $K(l_1, l_2)$. It is easy to check that the inequalities defining the fundamental domain are given by $2p_{1,k} \leq \sum_l p_{2,l}$ for all k and $2p_{2,l} \leq \sum_k p_{1,k}$ for all l.

Since, in particular, we have $d(\mathbf{P}_1, \mathbf{P}_2) \notin F(K(l_1, l_2))$ for some real root $d(\mathbf{P}_1, \mathbf{P}_2)$, we obtain that at least one inequality $2p_{1,k} > \sum_l p_{2,l}$ or $2p_{2,l} > \sum_k p_{1,k}$ holds. Moreover, it is well known that for every real root there exists a (up to isomorphism) unique indecomposable representation.

But even in the case $l_1 = 1$, i.e. the l_2-subspace quiver, it is not easy to decide if a root is a Schur root. One possibility is to consider the canonical decomposition of the root, see [17] for more details. If it consists of the dimension vector itself, the dimension vector is a Schur root. Nevertheless, it is possible to recursively construct plenty of real Schur roots of $K(l_1, l_2)$. Therefore, given a real Schur root $d(\mathbf{P}_1, \mathbf{P}_2)$ of $K(l_1, l_2)$ we consider the quiver $K(l_1, l_2 + 1)$. Then we have the following lemma:

Lemma 5.2 *The dimension vector $d(\mathbf{P}_1, \hat{\mathbf{P}}_2)$ with $\hat{\mathbf{P}}_2 = p_{2,1} + \ldots + p_{2,l_2} + (\sum_k p_{1,k})$ is a real Schur root for $K(l_1, l_2 + 1)$.*

Proof. It is easy to check that $\langle d(\mathbf{P}_1, \hat{\mathbf{P}}_2), d(\mathbf{P}_1, \hat{\mathbf{P}}_2) \rangle = 1$. Moreover, the canonical decomposition of $d(\mathbf{P}_1, \hat{\mathbf{P}}_2)$ is trivial because the one of $d(\mathbf{P}_1, \mathbf{P}_2)$ is trivial.

$\square$

Note that the setup in the proof is equivalent to the one of the generalized Kronecker quiver $K(\sum_k p_{1,k})$ with dimension vector $(1, \sum_k p_{1,k})$ which also is a real Schur root. Moreover, the preceding statement can be easily generalized to the case of general bipartite quivers.

For very small cases of l_1 and l_2, we have a complete description of all moduli spaces. More precisely, for $l_1 l_2 \leq 4$, the quiver $K(l_1, l_2)$ is of (extended) Dynkin type, and a classification of all isomorphism classes of representations is known:

If $l_1 l_2 \leq 3$, the group G has only finitely many orbits even in the space of all tuples of linear maps (the quiver $K(l_1, l_2)$ is then of Dynkin type A_2 or A_3 or D_4). If $l_1 l_2 = 4$, the quiver $K(l_1, l_2)$ is an extended Dynkin quiver of type $\widetilde{A}_3$ or $\widetilde{D}_4$, and again, a classification of all orbits is known. We use the following lemma which reduces the determination of all moduli spaces to the classification of indecomposable representations with trivial endomorphism ring:

Lemma 5.3 *We have $M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2) \neq \emptyset$ if and only if there exists a representation of $K(l_1, l_2)$ of dimension vector $d(\mathbf{P}_1, \mathbf{P}_2)$ with trivial endomorphism ring.*

Proof. By [17, Theorem 6.1], there exists a representation with trivial endomorphism ring of dimension vector $d = d(\mathbf{P}_1, \mathbf{P}_2)$ if and only if there exists a stable representation with respect to the stability $\Theta_d = \{d, \}$. But the conditions $\mu(e) < \mu(d)$, defined with respect to Θ_d and Θ, respectively, are equivalent.

$\square$

We can now apply the known representation theory of (extended) Dynkin quivers, and in particular the classification of representations with trivial endomorphism ring, to obtain the following description of pairs of ordered partitions (up to reordering) with nonempty moduli spaces:

$(l_1 = 1, l_2 = 1)$:
$$(1,0), \ (1,1), \ (0,1).$$

$(l_1 = 1, l_2 = 2)$:
$$(1, 0+0), \ (1, 1+0), \ (1, 1+1), \ (0, 1+0).$$

$(l_1 = 1, l_2 = 3)$:
$$(1, 0+0+0), \ (1, 1+0+0), \ (2, 1+1+1), \ (1, 1+1+0), \ (1, 1+1+1), \ (0, 1+0+0).$$

$(l_1 = 1, l_2 = 4)$:
$$(2i, i+i+i+(i+1)), \ (2i+1, (i+1)+(i+1)+(i+1)+(i+1)), \ (2i+1, i+(i+1)+(i+1)+(i+1)),$$
$$(1, 1+1+0+0), \ (2, 1+1+1+1),$$
$$(2i+2, i+(i+1)+(i+1)+(i+1)), \ (2i+1, i+i+i+i), \ (2i+1, i+i+i+(i+1)).$$

$(l_1 = 2, l_2 = 2)$:

$$(i + (i+1), i+i),\ (i+i, i+(i+1)),$$

$$(0+1, 0+1),\ (1+1, 1+1),$$

$$((i+1) + (i+1), i+(i+1)),\ (i+(i+1), (i+1)+(i+1)),\ \text{for } i \geq 0.$$

In almost all of the above cases, the moduli space $M^{\text{st}}(\mathbf{P}_1, \mathbf{P}_2)$ reduces to a single point, except the following:

$$M^{\text{st}}(2, 1+1+1+1) \simeq \mathbf{P}^1 \setminus \{0, 1, \infty\},$$

$$M^{\text{st}}(1+1, 1+1) \simeq \mathbf{P}^1 \setminus \{0, 1\}.$$

We now consider a particular case of the smooth models of the previous section.

Call a tuple $((f_{k,l}), w_l)$ consisting of a system of linear maps $(f_{k,l} : W_l \to V_k)_{k,l}$ as above and a system of vectors $(w_l \in W_l)_l$ stable if for all tuples $U_l \subset W_l$ of subspaces, we have $\sum_k \dim \sum_l f_{kl}(U_l) \geq \frac{a}{b} \sum_l \dim U_l$, with strict inequality if $w_l \in U_l$ for all $l = 1, \ldots, l_2$. Such tuples are considered up to simultaneous base change in all V_k and all W_l, i.e. tuples $(f_{k,l}, w_l)$ and $(f'_{k,l}, w'_l)$ are equivalent if there exist automorphisms $g_k \in \text{GL}(V_k)$ and $h_l \in \text{GL}(W_l)$ such that $f'_{k,l} = g_k f_{k,l} h_l^{-1}$ and $w'_l = h_l w_l$ for all k, l.

We define a dimension vector n^{b} for $K(l_1, l_2)$ by $n^{\text{b}}_{i_k} = 0$ for all $k = 1, \ldots, l_1$ and $n^{\text{b}}_{j_l} = 1$ for all $l = 1, \ldots, l_2$.

Theorem 5.4 *The moduli space*

$$M^{\text{b}}(\mathbf{P}_1, \mathbf{P}_2) = M^{\Theta}_{d(\mathbf{P}_1, \mathbf{P}_2), n^{\text{b}}}(K(l_1, l_2))$$

parametrizes equivalence classes of stable tuples $((f_{k,l}), w_k)$ as above.

We also have a dual version: we define n^{f} by $n^{\text{f}}_{i_k} = 1$ for all $k = 1, \ldots, l_1$ and $n^{\text{f}}_{j_l} = 0$ for all $l = 1, \ldots, l_2$. The corresponding moduli space

$$M^{\text{f}}(\mathbf{P}_1, \mathbf{P}_2) = M^{\Theta}_{d(\mathbf{P}_1, \mathbf{P}_2), n^{\text{f}}}(K(l_1, l_2))$$

parametrizes stable tuples $((f_{k,l}), v_k)$ consisting of a system of linear maps $(f_{k,l} : W_l \to V_k)_{k,l}$ as above and a system of vectors $(v_k \in V_k)_k$, where stable means that for all tuples $U_l \subset W_l$ of subspaces, we have $\sum_k \dim \sum_l f_{kl}(U_l) \geq \frac{a}{b} \sum_l \dim U_l$, with strict inequality if $v_k \in \sum_l f_{kl}(U_l)$ for all $k = 1, \ldots, l_1$.

We define the generating series of Euler characteristics of these moduli spaces by a slight variant of the series $Q^{(n)}_\mu(x)$ of the previous section, namely

$$F^{\text{b}}_{(a,b)} = \sum_{k \geq 0} \sum_{\substack{(\mathbf{P}_1, \mathbf{P}_2): \\ |\mathbf{P}_1| = ka, |\mathbf{P}_2| = kb}} \chi(M^{\text{b}}(\mathbf{P}_1, \mathbf{P}_2)) s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb} \in B$$

and

$$F^{\text{f}}_{(a,b)} = \sum_{k \geq 0} \sum_{\substack{(\mathbf{P}_1, \mathbf{P}_2): \\ |\mathbf{P}_1| = ka, |\mathbf{P}_2| = kb}} \chi(M^{\text{f}}(\mathbf{P}_1, \mathbf{P}_2)) s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb} \in B$$

6 Refined GW/Kronecker correspondence

The main result of this section is the following description of the series $f_{(a,b)}$ appearing in the factorization (1) in terms of Euler characteristics of the smooth models of complete bipartite quivers of the previous section:

Theorem 6.1 *For all coprime* (a, b), *we have*

$$f_{(a,b)} = (F_{(a,b)}^{\mathrm{b}})^{1/b} = (F_{(a,b)}^{\mathrm{f}})^{1/a}.$$

Proof. We consider the Poisson algebra $\overline{B} = B(K(l_1, l_2)) = \mathbb{Q}[[x_1, \ldots, x_{l_1}, y_1, \ldots, y_{l_2}]]$. It embeds into the algebra $B = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}][[s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2}]]$ of Section 2 by substituting $x_k = s_k x$ and $y_l = t_l y$.

The Poisson automorphism T_{i_k} of $\overline{B}$ is then given by

$$T_{i_k}(x_l) = x_l, \quad T_{i_k}(y_l) = y_l(1 + x_k),$$

and similarly T_{j_k} is given by

$$T_{j_k}(x_l) = x_l(1 + y_k)^{-1}, \quad T_{j_k}(y_l) = y_l.$$

Under the above substitution, this induces automorphisms of B given by

$$T_{i_k}(x) = x, \quad T_{i_k}(y) = y(1 + s_k x),$$

$$T_{j_k}(x) = x(1 + t_k y)^{-1}, \quad T_{j_k}(y) = y.$$

It follows that $T_i = \prod_{k=1}^{l_1} T_{i_k}$ and $T_j = \prod_{l=1}^{l_2} T_{j_l}$ are precisely the automorphisms $T_{(1,0), \prod_{k=1}^{l_1}(1+s_k x)}$ and $T_{(0,1), \prod_{l=1}^{l_2}(1+t_l y)}$, respectively, of Section 2.

Now Theorem 4.3 gives a factorization

$$T_i T_j = \prod_{\mu \in \mathbb{Q}}^{\leftarrow} T_\mu,$$

where T_μ is given as follows in terms of the generating functions $Q_\mu^{i_k}(x, y), Q_\mu^{j_l}(x, y)$:

$$T_\mu(x_{i_k}) = x_{i_k} \underbrace{\prod_{k'} Q_\mu^{i_{k'}}(x, y)^{\{i_{k'}, i_k\}}}_{=1} \prod_l Q_\mu^{j_l}(x, y)^{\{j_l, i_k\}},$$

$$T_\mu(x_{j_l}) = x_{j_l} \prod_k Q_\mu^{i_k}(x, y)^{\{i_k, j_l\}} \underbrace{\prod_{l'} Q_\mu^{j_{l'}}(x, y)^{\{j_{l'}, j_l\}}}_{=1},$$

the simplifications resulting from the calculation of $\{\_, \_\}$ above. It follows that T_μ induces the following automorphism of B:

$$T_\mu(x) = x \prod_l Q_\mu^{j_l}(x, y), \quad T_\mu(y) = y \prod_k Q_\mu^{i_k}(x, y).$$

We write $\mu = b/(a + b)$ for $a, b \in \mathbb{N}$ coprime. In the series $Q_\mu^{i_k}(x, y), Q_\mu^{j_l}(x, y)$, only monomials $x^{\mathbf{P}_1} y^{\mathbf{P}_2}$ with $|\mathbf{P}_1| = ka$, $|\mathbf{P}_2| = kb$ for some $k \geq 0$ appear (or, under the above substitution, only monomials $s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb}$ appear). Using Lemma 4.4, we have

$$(\prod_l Q_\mu^{j_l}(x, y))^a (\prod_k Q_\mu^{i_k}(x, y))^{-b} = 1.$$

Since a, b are coprime, we can choose $c, d \in \mathbb{Z}$ such that $ac + bd = 1$. We can then define

$$G_\mu(x, y) = (\prod_k Q_\mu^{i_k}(x, y))^c (\prod_l Q_\mu^{j_l}(x, y))^d.$$

A short calculation shows that

$$G_\mu(x, y)^a = \prod_k Q_\mu^{i_k}(x, y), \quad G_\mu(x, y)^b = \prod_l Q_\mu^{j_l}(x, y).$$

Thus, we have written the automorphism T_μ in the form

$$T_\mu(x) = x G_\mu(x, y)^{-b}, \quad T_\mu(y) = y G_\mu(x, y)^a,$$

i.e. $T_\mu = T_{(a,b), G_\mu(x,y)}$ is an element of the tropical vertex group H, and $G_\mu(x, y)$ equals the series $f_{(a,b)}$ by uniqueness of the factorization (1).

We have

$$G_\mu(x, y)^b = Q_\mu^{(n_0)}(x, y) = F_{(a,b)}^{\mathrm{b}}$$

(and similarly for $F_{(a,b)}^{\mathrm{f}}$), proving the theorem.

$\square$

Comparison of this theorem with Theorem 3.2 yields the first instance of the refined GW/Kronecker correspondence:

Corollary 6.2 *For all coprime (a, b), we have*

$$\exp \left(\sum_{k=1}^\infty \sum_{|\mathbf{P}_1|=ka, |\mathbf{P}_2|=kb} k N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb} \right) =$$

$$= \left(\sum_{k=0}^\infty \sum_{|\mathbf{P}_1|=ka, |\mathbf{P}_2|=kb} \chi(M^{\mathrm{b}}(\mathbf{P}_1, \mathbf{P}_2)) s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb} \right)^{1/b} =$$

$$= \left(\sum_{k=0}^\infty \sum_{|\mathbf{P}_1|=ka, |\mathbf{P}_2|=kb} \chi(M^{\mathrm{f}}(\mathbf{P}_1, \mathbf{P}_2)) s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb} \right)^{1/a} .$$

Comparing coefficients, we see that all Gromov-Witten invariants $N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$ are determined by all Euler characteristics $\chi(M^{\mathrm{b}}(\mathbf{P}_1, \mathbf{P}_2))$ (or $\chi(M^{\mathrm{f}}(\mathbf{P}_1, \mathbf{P}_2))$) and vice versa.

7 Derivation of functional equations

In this section, we apply Theorem 4.5 to derive functional equations determining the series $f_{(a,b)}$.

For a pair of ordered partitions $(\mathbf{P}_1, \mathbf{P}_2)$, we abbreviate by $\chi(\mathbf{P}_1, \mathbf{P}_2)$ the Euler characteristic $\chi(M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2))$.

Theorem 7.1 *For coprime (a, b), the series $f_{(a,b)}$ is given by*

$$f_{(a,b)} = \prod_{k \geq 1} \prod_{\substack{|\mathbf{P}_1| = ka \\ |\mathbf{P}_2| = kb}} (R^{\mathbf{P}_1, \mathbf{P}_2})^{k\chi(\mathbf{P}_1, \mathbf{P}_2)},$$

where the series $R^{\mathbf{P}_1, \mathbf{P}_2} \in B$ are determined by the following system of functional equations:
For all pairs of ordered partitions $(\mathbf{P}_1, \mathbf{P}_2)$ as above,

$$R^{\mathbf{P}_1, \mathbf{P}_2} = (1 - s^{\mathbf{P}_1} t^{\mathbf{P}_2} (x^a y^b)^k \prod_{k' \geq 1} \prod_{\substack{|\mathbf{P}_1'| = k'a \\ |\mathbf{P}_2'| = k'b}} (R^{\mathbf{P}_1', \mathbf{P}_2'})^{-\langle d(\mathbf{P}_1, \mathbf{P}_2), d(\mathbf{P}_1', \mathbf{P}_2') \rangle \chi(\mathbf{P}_1', \mathbf{P}_2')})^{-1}.$$

Proof. In the notation of the proof of Theorem 6.1, we have

$$f_{(a,b)} = G_\mu(x, y) = \prod_k Q_\mu^{i_k}(x, y))^c \prod_l Q_\mu^{j_l}(x, y))^d = Q_\mu^{n(c,d)\cdot}(x, y),$$

where $n(c, d)\cdot$ is the functional given by $n(c, d) \cdot i_k = c$ for $k = 1, \ldots, l_1$ and $n(c, d) \cdot j_l = d$ for $l = 1, \ldots, l_2$. We then have

$$n(c, d) \cdot d(\mathbf{P}_1, \mathbf{P}_2) = c|\mathbf{P}_1| + d|\mathbf{P}_2| = kac + kbd = k.$$

Now the statement of the theorem is just an adaption of Theorem 4.5 to the present notation.

$\square$

This second instance of the refined GW/Kronecker correspondence shows (by comparing coefficients $(x^a y^b)^k$) that all Gromov-Witten invariants $N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$ are determined by all Euler characteristics $\chi(\mathbf{P}_1, \mathbf{P}_2)$ and vice versa, although involving an infinite system of coupled functional equations. To extract more direct information on the relation between these two geometries, we restrict to more particular cases in the following sections.

8 Specialization

The system of functional equations in Theorem 7.1, as well as some of the special cases considered in the following sections, simplify considerably once we specialize all variables s_k and t_l to one variable t. We denote by $N_{(a,b)}[k]$ the sum

$$N_{(a,b)}[k] = \sum_{|\mathbf{P}_1| = ka, |\mathbf{P}_2| = kb} N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$$

of Gromov-Witten invariants and by $\chi(k)$ the corresponding sum

$$\chi_{(a,b)}(k) = \sum_{|\mathbf{P}_1| = ka, |\mathbf{P}_2| = kb} \chi(\mathbf{P}_1, \mathbf{P}_2)$$

of Euler characteristics. We denote by $f_{(a,b)}(t) \in \mathbb{Q}[x^a y^b][[t]]$ the specialization of the series $f_{(a,b)}$ (and similarly $R^{\mathbf{P}_1, \mathbf{P}_2}(t)$) and define

$$E = \frac{l_1 l_2 ab - l_2 a^2 - l_1 b^2}{l_1 l_2} \in \mathbb{Q}.$$

Theorem 8.1 *The series $f_{(a,b)}(t)$ is determined by the single functional equation*

$$f_{(a,b)}(t) = \prod_{k \geq 1}(1 - ((tx)^a(ty)^b f_{(a,b)}(t)^E)^k)^{-k\chi_{(a,b)}(k)}.$$

Proof. Combining the functional equations of Theorem 7.1, we have $f_{(a,b)}(t) =$

$$\prod_{\substack{k \geq 1 \\ |\mathbf{P}_1|=ka \\ |\mathbf{P}_2|=kb}} (1 - (tx)^{ka}(ty)^{kb} \prod_{\substack{k' \geq 1 \\ |\mathbf{P}'_1|=k'a \\ |\mathbf{P}'_2|=k'b}} R^{\mathbf{P}'_1,\mathbf{P}'_2}(t)^{-\langle d(\mathbf{P}_1,\mathbf{P}_2),d(\mathbf{P}'_1,\mathbf{P}'_2)\rangle \chi(\mathbf{P}'_1,\mathbf{P}'_2)})^{-k\chi(\mathbf{P}_1,\mathbf{P}_2)}.$$

We have to study the inner product

$$\prod_{\substack{|\mathbf{P}'_1|=k'a \\ |\mathbf{P}'_2|=k'b}} R^{\mathbf{P}'_1,\mathbf{P}'_2}(t)^{-\langle d(\mathbf{P}_1,\mathbf{P}_2),d(\mathbf{P}'_1,\mathbf{P}'_2)\rangle \chi(\mathbf{P}'_1,\mathbf{P}'_2)}$$

on the right hand side.

Using the $S_{l_1} \times S_{l_2}$-symmetry of the quiver, we see that both the series $R^{\mathbf{P}_1,\mathbf{P}_2}(t)$ and the Euler characteristic $\chi(\mathbf{P}_1,\mathbf{P}_2)$ are invariant under permutations. Thus, in the above product over (pairs of) ordered partitions, it suffices to multiply contributions from ordered partitions $\mathbf{P}$ satisfying $p_1 \geq \ldots \geq p_l$, and to take multiplicities into account. This allows us to simplify the Euler form:

Let $\mathbf{P}^0$ be an ordered partition of length l satisfying $p_1^0 \geq \ldots \geq p_l^0$, and denote by $z(\mathbf{P}^0)$ the number of rearrangements of $\mathbf{P}^0$ into an ordered partition $\mathbf{P}' \models \mathbf{P}^0$. For an ordered partition $\mathbf{P}$ of length l, we then have

$$\sum_{\mathbf{P}' \models \mathbf{P}^0} \sum_k p_k p'_k = \frac{z(\mathbf{P}^0)}{l!} \sum_{\sigma \in S_l} \sum_k p_k p^0_{\sigma(k)} =$$

$$= \frac{z(\mathbf{P}^0)}{l!} \sum_k p_k (\sum_{\sigma \in S_l} p^0_{\sigma(k)}) = \frac{z(\mathbf{P}^0)}{l!}(l-1)! \sum_{k,k'} p_k p^0_{k'} = z(\mathbf{P}^0)\frac{1}{l}|\mathbf{P}||\mathbf{P}^0|.$$

This implies the following identity for the Euler form:

$$\sum_{\mathbf{P}'_1 \models \mathbf{P}_1^0, \mathbf{P}'_2 \models \mathbf{P}_2^0} \langle d(\mathbf{P}_1,\mathbf{P}_2), d(\mathbf{P}'_1,\mathbf{P}'_2)\rangle = z(\mathbf{P}_1^0)z(\mathbf{P}_2^0)kk'\frac{l_2 a^2 + l_1 b^2 - l_1 l_2 ab}{l_1 l_2},$$

which is just $-z(\mathbf{P}_1^0)z(\mathbf{P}_2^0)kk'E$. Using this identity, the inner product above simplifies as follows ($\mathbf{P}_1^0$, $\mathbf{P}_2^0$ denoting weakly descending ordered partitions as before):

$$\prod_{\substack{|\mathbf{P}'_1|=k'a \\ |\mathbf{P}'_2|=k'b}} R^{\mathbf{P}'_1,\mathbf{P}'_2}(t)^{-\langle d(\mathbf{P}_1,\mathbf{P}_2),d(\mathbf{P}'_1,\mathbf{P}'_2)\rangle \chi(\mathbf{P}'_1,\mathbf{P}'_2)} =$$

$$= \prod_{\substack{|\mathbf{P}_1^0|=k'a \\ |\mathbf{P}_2^0|=k'b}} \prod_{\substack{\mathbf{P}'_1 \models \mathbf{P}_1^0 \\ \mathbf{P}'_2 \models \mathbf{P}_2^0}} R^{\mathbf{P}_1^0,\mathbf{P}_2^0}(t)^{-\langle d(\mathbf{P}_1,\mathbf{P}_2),d(\mathbf{P}'_1,\mathbf{P}'_2)\rangle \chi(\mathbf{P}_1^0,\mathbf{P}_2^0)} =$$

$$= \prod_{\substack{|\mathbf{P}_1^0|=k'a \\ |\mathbf{P}_2^0|=k'b}} R^{\mathbf{P}_1^0,\mathbf{P}_2^0}(t)^{-\sum_{\mathbf{P}'_1 \models \mathbf{P}_1^0, \mathbf{P}'_2 \models \mathbf{P}_2^0}\langle d(\mathbf{P}_1,\mathbf{P}_2),d(\mathbf{P}'_1,\mathbf{P}'_2)\rangle \chi(\mathbf{P}_1^0,\mathbf{P}_2^0)} =$$

$$= \prod_{\substack{|\mathbf{P}_1^0|=k'a \\ |\mathbf{P}_2^0|=k'b}} R^{\mathbf{P}_1^0, \mathbf{P}_2^0}(t)^{z(\mathbf{P}_1^0)z(\mathbf{P}_2^0)kk'E\chi(\mathbf{P}_1', \mathbf{P}_2')} =$$

$$= \prod_{\substack{|\mathbf{P}_1'|=k'a \\ |\mathbf{P}_2'|=k'b}} R^{\mathbf{P}_1', \mathbf{P}_2'}(t)^{kk'E\chi(\mathbf{P}_1', \mathbf{P}_2')}.$$

Rewriting the above expression for $f_{(a,b)}(t)$ using this, we get $f_{(a,b)}(t) =$

$$\prod_{k \geq 1} \prod_{\substack{|\mathbf{P}_1|=ka \\ |\mathbf{P}_2|=kb}} (1 - (tx)^{ka}(ty)^{kb} \prod_{k' \geq 1} \prod_{\substack{|\mathbf{P}_1'|=k'a \\ |\mathbf{P}_2'|=k'b}} R^{\mathbf{P}_1', \mathbf{P}_2'}(t)^{kk'E\chi(\mathbf{P}_1', \mathbf{P}_2')})^{-k\chi(\mathbf{P}_1, \mathbf{P}_2)}.$$

We identify the original product expansion for the series $f_{(a,b)}(t)$ in the inner product, which yields the claimed functional equation.

$\square$

We can now give a formula for the specialized Gromov-Witten invariants $N_{(a,b)}[k]$ applying the methods of [12] to the functional equation of Theorem 8.1:

Corollary 8.2 *We have*

$$N_{(a,b)}[k] = \frac{1}{Ek^2} \sum_{\mathbf{r}} \prod_i \binom{Eki\chi_{(a,b)}(i) + r_i - 1}{r_i},$$

the sum running over all ordered partitions $\mathbf{r} = r_1 + \ldots$ such that $\sum_i ir_i = k$.

Proof. We apply [12, Proposition 4.4] to the series $F = f_{(a,b)}(t)^E$. Unwinding the definitions, the formula follows.

$\square$

9 Coprime case

As a first step towards extracting explicit formulas out of the functional equations of Theorem 7.1 (for non-specialized variables), we can compare the coefficients of $x^a y^b$ in the equation of Theorem 6.1 as well as in the functional equations of Theorem 7.1 to get:

Corollary 9.1 *For coprime (a, b) and a pair of ordered partitions $(\mathbf{P}_1, \mathbf{P}_2)$ such that $|\mathbf{P}_1| = a$ and $|\mathbf{P}_2| = b$, we have*

$$N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] = \frac{1}{b} \cdot \chi(M^\mathbf{b}(\mathbf{P}_1, \mathbf{P}_2)) = \frac{1}{a} \cdot \chi(M^\mathbf{f}(\mathbf{P}_1, \mathbf{P}_2))$$

and

$$N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] = \chi(M^\mathrm{st}(\mathbf{P}_1, \mathbf{P}_2)).$$

For general coprime (a, b), the Euler characteristic $\chi(\mathbf{P}_1, \mathbf{P}_2)$ can be computed (preferably with computer aid) by working out Theorem 4.1 for the quiver $K(l_1, l_2)$ to obtain the Poincaré polynomial, and then specializing $q = 1$:

Theorem 9.2 *For a pair $(\mathbf{P}_1, \mathbf{P}_2)$ of ordered partitions such that $|\mathbf{P}_1| = a$ and $|\mathbf{P}_2| = b$ are coprime, we have $\sum_i \dim H^i(M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2), \mathbb{Q})q^{i/2} = (q-1) \times$*

$$\times \sum (-1)^{s-1} q^{\sum_{r \leq s}(a_r b_s - \sum_k p_{1,k}^r p_{1,k}^s - \sum_l p_{2,l}^r p_{2,l}^s)} \prod_{r=1}^t (\prod_k \prod_{j=1}^{p_{1,k}^r}(1 - q^{-j}) \prod_l \prod_{j=1}^{p_{2,l}^r}(1 - q^{-j}))^{-1},$$

where the sum runs over all decompositions $\mathbf{P}_i = \mathbf{P}_i^{(1)} + \ldots + \mathbf{P}_i^{(t)}$ for $i = 1, 2$ into ordered partitions $\mathbf{P}_i^{(r)} = p_{i,1}^r + \ldots + p_{i,l_i}^r$ such that for $a_r = |\mathbf{P}_1^{(r)}|$ and $b_r = |\mathbf{P}_2^{(r)}|$, we have $(a_r, b_r) \neq (0,0)$ for all $r = 1, \ldots, t$ and

$$\frac{b_1 + \ldots + b_r}{a_1 + \ldots + a_r} > \frac{b}{a}$$

for all $r < t$.

Remark 9.3

- In the present context, this formula seems to be somewhat tautological: the factorization formula Theorem 4.3 ultimately follows from the Harder-Narasimhan recursion for the moduli spaces $M_d^{\Theta-\mathrm{st}}(Q)$, and the formula in Theorem 4.1 is a resolution of the very same recursion.

  Also note the disadvantage that the full Poincaré polynomial has to be computed to extract just the Euler characteristic, since every individual summand in the above sum has a pole at $q = 1$.

As a particular example, localization techniques allow to extract the following explicit formula in specialized variables (see Theorem 15.3):

Theorem 9.4 *For arbitrary d, we have*

$$N_{(d,d-1)}[1] = \chi_{(d,d-1)}(1) = \frac{l_1 l_2}{d((l_1-1)d+1)} \binom{(l_1-1)(l_2-1)d + l_2 - 1}{d-1}.$$

Another particular example is $N_{(3,5)}[1] = 204$ for $l_1 = 3 = l_2$, which is worked out in detail in Example 17.4.

10 Small length

To work out all series $f_{(a,b)}$ in the case where $l_1 l_2 \leq 4$, we first note the following trivial case of the functional equations of Theorem 7.1:

Lemma 10.1 *Suppose that the following holds for coprime a and b:*

- *For all pairs of ordered partitions $(\mathbf{P}_1, \mathbf{P}_2)$ such that $|\mathbf{P}_1| = a$ and $|\mathbf{P}_2| = b$, the moduli space $M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2)$ is either empty or a single point. Let $((\mathbf{P}_1^{(r)}, \mathbf{P}_2^{(r)}))_r$ be a complete list of those where the latter holds.*

- *The Euler form fulfills $\langle d(\mathbf{P}_1^{(r)}, \mathbf{P}_2^{(r)}), d(\mathbf{P}_1^{(s)}, \mathbf{P}_2^{(s)}) \rangle = \delta_{r,s}$.*

- *For all pairs of ordered partitions $(\mathbf{P}_1, \mathbf{P}_2)$ such that $|\mathbf{P}_1| = ka$ and $|\mathbf{P}_2| = kb$ for some $k \geq 2$, the moduli space $M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2)$ is empty.*

Then

$$f_{(a,b)} = \prod_r (1 + s^{\mathbf{P}_1^{(r)}} t^{\mathbf{P}_2^{(r)}} x^a y^b),$$

and thus

$$N_{(ka,kb)}[(k\mathbf{P}_1^{(r)}, k\mathbf{P}_2^{(r)})] = \frac{(-1)^{k-1}}{k^2}.$$

Proof. By Theorem 7.1, we have $f_{(a,b)} = \prod_r R^{\mathbf{P}_1^{(r)},\mathbf{P}_2^{(r)}}$ and

$$R^{\mathbf{P}_1^{(r)},\mathbf{P}_2^{(r)}} = (1 - s^{\mathbf{P}_1^{(r)}} t^{\mathbf{P}_2^{(r)}} x^a y^b (R^{\mathbf{P}_1^{(r)},\mathbf{P}_2^{(r)}})^{-1})^{-1}.$$

The solution to this latter equation is evidently $1 + s^{\mathbf{P}_1^{(r)}} t^{\mathbf{P}_2^{(r)}} x^a y^b$.

$\square$

To complete the description of the series $f_{(a,b)}$ when $l_1 l_2 \leq 4$, there are thus only two remaining cases to consider:

We first consider the case $l_1 = 2 = l_2$ and the slope $(a, b) = (1, 1)$. We abbreviate $R^{(1+0,1+0)}$ by R_{11} and define R_{12}, R_{21} and R_{22} similarly. After computing the relevant values of the Euler form, we find

$$f_{(1,1)} = R_{11} R_{12} R_{21} R_{22},$$

where

$$R_{11} = (1 - s_1 t_1 xy \frac{R_{22}}{R_{11}})^{-1}, \ \ R_{22} = (1 - s_2 t_2 xy \frac{R_{11}}{R_{22}})^{-1},$$

and thus

$$R_{11} = \frac{1 + s_1 t_1 xy}{1 - s_1 s_2 t_1 t_2 x^2 y^2}$$

(and similarly for R_{12}, R_{21} and R_{22}), thus

$$f_{(1,1)} = \frac{(1 + s_1 t_1 xy)(1 + s_1 t_2 xy)(1 + s_2 t_1 xy)(1 + s_2 t_2 xy)}{(1 - s_1 s_2 t_1 t_2 x^2 y^2)^4}.$$

This gives

$$N_{(1,1)}[(k + 0, k + 0)] = \frac{(-1)^{k-1}}{k^2}$$

and

$$N_{(1,1)}[(k + k, k + k)] = \frac{2}{k^2}.$$

Similarly, we treat the case $l_1 = 1$, $l_2 = 4$ and the slope $(a, b) = (1, 2)$: we abbreviate $R^{(1,1+1+0+0)}$ by R_{12} and similarly for R_{13}, R_{14}, R_{23}, R_{24} and R_{34}. Then we have

$$f_{(1,2)} = R_{12} R_{13} R_{14} R_{23} R_{24} R_{34} (R^{(2,1+1+1+1)})^{-2}.$$

We have the equation

$$R^{(2,1+1+1+1)} = (1 - s^2 t_1 t_2 t_3 t_4 x^2 y^4)^{-1}$$

and

$$R_{12} = (1 - s t_1 t_2 xy^2 \frac{R_{34}}{R_{12}})^{-1}$$

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with solution

$$R_{12} = \frac{1 + st_1 t_2 xy^2}{(1 - s^2 t_1 t_2 t_3 t_4 x^2 y^4)^2}$$

as above (and similarly for R_{13}, R_{14}, R_{23}, R_{24} and R_{34}). This yields the result

$$f_{(1,2)} = \frac{\prod_{1 \leq i < j \leq 4}(1 + st_i t_j xy^2)}{(1 - s^2 t_1 t_2 t_3 t_4 x^2 y^4)^4}.$$

Again, this gives

$$N_{(1,2)}[(k, k + k + 0 + 0)] = \frac{(-1)^{k-1}}{k^2}$$

and

$$N_{(1,2)}[(2k, k + k + k + k)] = \frac{2}{k^2}.$$

11 Central slope

Now we specialize Theorem 7.1 to the central slope $a = 1 = b$; on the Gromov-Witten side, we are thus considering maps to $\mathbf{P}^2$ instead of an arbitrary weighted projective plane. We then consider pairs of ordered partitions $\mathbf{P}_1$, $\mathbf{P}_2$ such that $|\mathbf{P}_1| = k = |\mathbf{P}_2|$. In this case, localization techniques allow us to derive (see Corollary 16.4) that $\chi(\mathbf{P}_1, \mathbf{P}_2) = 0$ as soon as $k \geq 2$. For $k = 1$, the choice of $\mathbf{P}_1$ (resp. $\mathbf{P}_2$) is just the choice of an index $k = 1, \ldots, l_1$ (resp. $l = 1, \ldots, l_2$), and the resulting moduli spaces are single points, thus $\chi(\mathbf{P}_1, \mathbf{P}_2) = 1$. We denote the corresponding dimension vector by $d(k, l)$. The Euler form evaluates to

$$\langle d(k, l), d(k', l') \rangle = \delta_{k,k'} + \delta_{l,l'} - 1.$$

Furthermore, we have

$$E = \frac{l_1 l_2 - l_1 - l_2}{l_1 l_2}$$

in this case. The functional equations of Theorem 7.1 then simplify drastically and become algebraic:

Theorem 11.1 *The series $f_{(1,1)}$ is given as*

$$f_{(1,1)} = \prod_{k=1}^{l_1} \prod_{l=1}^{l_2} R^{k,l},$$

where the series $R^{k,l}$ are determined by the system of functional equations

$$R^{k,l} = 1 + s_k t_l xy \prod_{k' \neq k} \prod_{l' \neq l} R^{k',l'}.$$

This system of functional equations is reminiscent of the Q-systems of [8] and can be solved by multivariate Lagrange inversion (but the resulting formulas are not particularly explicit).

Specializing all variables s_k and t_l to one variable t and defining

$$f_{(1,1)}(t) = H(t)^{l_1 l_2},$$

the series H is determined by the single functional equation

$$H(t) = (1 - t^2 xy H(t)^{l_1 l_2 - l_1 - l_2})^{-1}.$$

But then it follows immediately from [10, Theorem 1.4] that:

Corollary 11.2 *We have*

$$f_{(1,1)}(t) = \left(\sum_{k \geq 0} \frac{1}{(l_1 l_2 - l_1 - l_2)k + 1} \binom{(l_1 - 1)(l_2 - 1)k}{k} (t^2 xy)^k \right)^{l_1 l_2},$$

confirming [2, Conjecture 1.4].

Specializing Corollary 8.2, we find

Corollary 11.3 *We have*

$$N_{(1,1)}[k] = \frac{l_1 l_2}{k^2} \binom{(l_1 - 1)(l_2 - 1)k - 1}{k - 1}.$$

In this special case, we can also confirm (a variant of) the integrality conjecture [3, Conjecture 6.2]:

Corollary 11.4 *We have*

$$\sum_{d | k} \mu(\frac{k}{d})(-1)^{(l_1 l_2 - l_1 - l_2)(d-k)} \frac{d^2}{k^2} N_{(1,1)}[d] \in \mathbf{N}.$$

Proof. By the previous corollary, the above Moebius inversion equals

$$\sum_{d | k} \mu(\frac{k}{d})(-1)^{(l_1 l_2 - l_1 - l_2)(d-k)} \binom{(l_1 - 1)(l_2 - 1)d - 1}{d - 1},$$

which is a nonnegative integer by [14, Theorem 3.2].

$\square$

12 Balanced case

In this section, we consider the case $l_1 = m = l_2$. We can then relate the geometry of the moduli spaces $M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2)$ to moduli spaces of representations of the m-Kronecker quiver.

For coprime a, b as above and $k \geq 1$, we consider m tuples $(f_k : W \to V)_k$ of linear maps from a kb-dimensional vector space W to a ka-dimensional one V, up to the base change action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$. We call such a tuple of linear maps stable if $\dim \sum_k f_k(U) > \frac{a}{b} \dim U$ for all non-zero proper subspaces U of W. There exists a moduli space $M^{\mathrm{st}}(ka, kb)$ parametrizing stable tuples up to base change.
From Theorem 15.2, we derive

Theorem 12.1 *We have*

$$\sum_{|\mathbf{P}_1| = ka, \, |\mathbf{P}_2| = kb} \chi(M^{\mathrm{st}}(\mathbf{P}_1, \mathbf{P}_2)) = m \cdot \chi(M^{\mathrm{st}}(ka, kb)).$$

This allows us to apply [12, Theorem 5.1] to confirm (again, a variant of) [3, Conjecture 6.2] in the balanced case and for specialized variables, as already indicated in [2]:

Corollary 12.2 *If $l_1 = m = l_2$, every specialized series $f_{(a,b)}(t)$ admits a product factorization*

$$f_{(a,b)}(t) = \prod_{k \geq 1} (1 - ((-1)^{mab-a^2-b^2} t)^k)^{-kd(a,b,k)}$$

for integral $d(a,b,k)$.

Proof. Applying Theorem 8.1 and using the above theorem, the series $f_{(a,b)}(t)^{1/m}$ is determined by the functional equation

$$f_{(a,b)}(t)^{1/m} = \prod_{k \geq 1} (1 - ((tx)^a (ty)^b (f_{(a,b)}^{1/m})^{mab-a^2-b^2})^k)^{-k\chi(M^{\mathrm{st}}(ka,kb))}.$$

Applying [12, Theorem 4.9], the statement follows.

$\square$

13 General commutator formula

In this section, we consider a more general class of bipartite quivers to obtain a partial GW/Kronecker correspondence for the commutator formula [3, Theorem 5.6], which on the Gromow-Witten side involves orbifold blow-ups. However, it will turn out that the correspondence is weaker than in the cases considered before. The following derivation of the correspondence follows the steps of Section 6 closely, thus some details will be omitted.

Fix tuples of nonnegative integers $l_i^* = (l_i^1, \ldots, l_i^{d_i})$ for $i = 1, 2$. Define a quiver $K(l_1^*, l_2^*)$ as follows: $K(l_1^*, l_2^*)$ has vertices i_ζ^r for $r = 1, \ldots, d_1$ and $\zeta = 1, \ldots, l_1^r$ and j_ξ^s for $s = 1, \ldots, d_2$ and $\xi = 1, \ldots, l_2^s$. There are rs arrows from each j_ξ^s to each i_ζ^r.

The quiver $K(l_1^*, l_2^*)$ is therefore bipartite with a "level structure", the vertices i_ζ^r and j_ξ^s being of level r and s, respectively, such that the number of arrows is given by the product of levels.

The quiver $K(l_1^*, l_2^*)$ has a natural $\prod_{r=1}^{d_1} S_{l_1^r} \times \prod_{s=1}^{d_2} S_{l_2^s}$-symmetry permuting vertices i (resp. j) of the same level.

Computing the antisymmetrized Euler form of $K(l_1^*, l_2^*)$, we get

$$\{i_{\zeta'}^{r'}, i_\zeta^r\} = 0, \quad \{j_\xi^s, i_\zeta^r\} = -rs, \quad \{i_\zeta^r, j_\xi^s\} = rs, \quad \{j_{\xi'}^{s'}, j_\xi^s\} = 0$$

for all $r, r', s, s', \zeta, \zeta', \xi, \xi'$.

We view dimension vectors for $K(l_1^*, l_2^*)$ as pairs of graded partitions (in the notation of [3, Section 5.5]) $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ as follows:

For $i = 1, 2$, the graded partition $\mathbf{G}_i$ is a tuple $\mathbf{G}_i = (\mathbf{P}_i^1, \ldots, \mathbf{P}_i^{d_i})$ of unordered partitions with all parts of $\mathbf{P}_i^r$ divisible by r (resp. s), that is,

$$\mathbf{P}_1^r = rd_{i_1^r} + \ldots + rd_{i_{l_1^r}^r}, \quad \mathbf{P}_2^s = sd_{j_1^s} + \ldots + sd_{j_{l_2^s}^s}.$$

The collection $(d_{i_\zeta^r}, d_{j_\xi^s})$ defines a dimension vector for $K(l_1^*, l_2^*)$. The size of $\mathbf{G}_i$ is defined as $|\mathbf{G}_i| = \sum_i |\mathbf{P}^i|$.

Specializing the results of Section 4 to this quiver, we find the following:

The algebra B is given by $B = \mathbb{Q}[[x_{i_\zeta^r}, x_{j_\xi^s}]]$; to simplify notation, we rename these variables as $x_\zeta^r = x_{i_\zeta^r}$, $y_\xi^s = x_{j_\xi^s}$. The Poisson automorphisms $T_{i_\zeta^r}$ and $T_{j_\xi^s}$ of B are given as follows:

$$T_{i_\zeta^r}(x_{\zeta'}^{r'}) = x_{\zeta'}^{r'}, \quad T_{i_\zeta^r}(y_\xi^s) = y_\xi^s(1 + x_\zeta^r)^{rs},$$

$$T_{j_\xi^s}(x_\zeta^r) = x_\zeta^r(1 + y_\xi^s)^{-rs}, \quad T_{j_\xi^s}(y_{\xi'}^{s'}) = y_{\xi'}^{s'}.$$

We consider B as a subalgebra of the algebra $B' = \mathbb{Q}[x, x^{-1}, y, y^{-1}][[s_\zeta^r, t_\xi^s]]$ via the identifications $x_\zeta^r = s_\zeta^r x^r$ and $y_\xi^s = t_\xi^s y^s$. Then the automorphisms $T_{i_\zeta^r}$, $T_{j_\xi^s}$ lift to the following $\mathbb{Q}[[s_\zeta^r, t_\xi^s]]$-linear automorphisms of B':

$$T_{i_\zeta^r}(x) = x, \quad T_{i_\zeta^r}(y) = y(1 + s_\zeta^r x^r)^r,$$

$$T_{j_\xi^s}(x) = x(1 + t_\xi^s y^s)^{-s}, \quad T_{j_\xi^s}(y) = y.$$

We define

$$T_i = \prod_{r=1}^{d_1} \prod_{\zeta=1}^{l_1^r} T_{i_\zeta^r}, \quad T_j = \prod_{s=1}^{d_2} \prod_{\xi=1}^{l_2^s} T_{j_\xi^s},$$

thus

$$T_i(x) = x, \quad T_i(y) = y \prod_{r=1}^{d_1} \prod_{\zeta=1}^{l_1^r} (1 + s_\zeta^r x^r)^r,$$

$$T_j(x) = x \prod_{s=1}^{d_2} \prod_{\xi=1}^{l_2^s} (1 + t_\xi^s y^s)^{-s}, \quad T_j(y) = y.$$

We define a (Θ, κ)-stability on $K(l_1^*, l_2^*)$ by

$$\Theta_{i_\zeta^r} = 0, \quad \Theta_{j_\xi^s} = s,$$

$$\kappa_{i_\zeta^r} = r, \quad \kappa_{j_\xi^s} = s.$$

Given coprime integers a, b, we then have the following for a dimension vector $\mathbf{d} = (d_{i_\zeta^r}, d_{j_\xi^s})$ of $K(l_1^*, l_2^*)$:

$\mu(\mathbf{d}) = \frac{b}{a+b}$ if and only if $\sum_r \sum_\zeta r d_{i_\zeta^r} = ka$, $\sum_s \sum_\xi s d_{j_\xi^s} = kb$ for some $k \geq 1$, or, in other words, if and only if $(|\mathbf{G}_1|, |\mathbf{G}_2|)$ is a multiple of (a, b).

By Theorem 4.3, we have

$$T_i T_j = \prod_{\mu \in \mathbb{Q}}^{\leftarrow} T_\mu$$

for automorphisms T_μ of B given as follows:

$$T_\mu(x_\zeta^r) = x_\zeta^r \prod_{r'} \prod_{\zeta'} (Q_\mu^{i_{\zeta'}^{r'}})^{\{i_{\zeta'}^{r'}, i_\zeta^r\}} \prod_s \prod_\xi (Q_\mu^{j_\xi^s})^{\{j_\xi^s, i_\zeta^r\}},$$

$$T_\mu(y_\xi^s) = y_\xi^s \prod_r \prod_\zeta (Q_\mu^{i_\zeta^r})^{\{i_\zeta^r, j_\xi^s\}} \prod_{s'} \prod_{\xi'} (Q_\mu^{j_{\xi'}^{s'}})^{\{j_{\xi'}^{s'}, j_\xi^s\}}.$$

Using the computation of the antisymmetrized Euler form of $K(l_1^*, l_2^*)$ above, this simplifies to

$$T_\mu(x_\zeta^r) = x_\zeta^r \prod_s \prod_\xi (Q_\mu^{j_\xi^s})^{-rs},$$

$$T_\mu(y^s_\xi) = y^s_\xi \prod_r \prod_\zeta (Q^{i^r_\zeta}_\mu)^{rs}.$$

Using this description, we can see that T_μ extends to an automorphism of B', namely

$$T_\mu(x) = x \prod_s \prod_\xi (Q^{j^s_\xi}_\mu)^{-s},$$

$$T_\mu(y) = y \prod_r \prod_\zeta (Q^{i^r_\zeta}_\mu)^r.$$

The analogue of Lemma 4.4 for the more general stability (Θ, κ) yields

$$1 = Q^{\Theta - \mu\kappa}_\mu = \prod_r \prod_\zeta (Q^{i^t_\zeta}_\mu)^{-\frac{b}{a+b}r} \prod_s \prod_\xi (Q^{j^s_\xi}_\mu)^{\frac{a}{a+b}s},$$

and thus

$$1 = \prod_r \prod_\zeta (Q^{i^r_\zeta}_\mu)^{-br} \prod_s \prod_\xi (Q^{j^s_\xi}_\mu)^{as}.$$

By coprimality of a, b, we can choose c, d such that $ac + bd = 1$. We define

$$F_\mu = \prod_r \prod_\zeta (Q^{i^r_\zeta}_\mu)^{cr} \prod_s \prod_\xi (Q^{j^s_\xi}_\mu)^{ds}.$$

Then we have

$$F^a_\mu = \prod_r \prod_\zeta (Q^{i^r_\zeta}_\mu)^r, \quad F^{-b}_\mu = \prod_s \prod_\xi (Q^{j^s_\xi}_\mu)^{-s},$$

thus

$$T_\mu(x) = xF^{-b}_\mu, \quad T_\mu(y) = yF^a_\mu$$

is an element of the tropical vertex H.

The monomials in the variables x^r_ζ, y^s_ξ appearing in the series Q_μ, F_μ are the

$$\prod_r \prod_\zeta (x^r_\zeta)^{d_{i^r_\zeta}} \prod_s \prod_\xi (y^s_\xi)^{d_{j^s_\xi}}$$

such that

$$\sum_r \sum_\zeta r d_{i^r_\zeta} = ka, \quad \sum_s \sum_\xi s d_{j^s_\xi} = kb$$

for some $k \geq 1$. After embedding into B', these monomials are identified with

$$\prod_r \prod_\zeta ((s^r_\zeta)^{d_{i^r_\zeta}} x^{r d_{i^r_\zeta}}) \prod_s \prod_\xi ((t^s_\xi)^{d_{j^s_\xi}} y^{s d_{j^s_\xi}}) =$$

$$= s^{\mathbf{G}_1} t^{\mathbf{G}_2} x^{|\mathbf{G}_1|} y^{|\mathbf{G}_2|} = s^{\mathbf{G}_1} t^{\mathbf{G}_2} (x^a y^b)^k$$

in the notation of [3, Section 5.6].

We consider the moduli space

$$M^{\mathrm{b}}(\mathbf{G}_1, \mathbf{G}_2) = M^{(\Theta, \kappa)}_{d(\mathbf{G}_1, \mathbf{G}_2), n^{\mathrm{b}}}(K(l^*_1, l^*_2))$$

for $d(\mathbf{G}_1, \mathbf{G}_2)_{i^r_\zeta} = \frac{1}{r} p^r_{1,\zeta}$, $d(\mathbf{G}_1, \mathbf{G}_2)_{j^s_\xi} = \frac{1}{s} p^s_{2,\xi}$, $n^{\mathrm{b}}_{i^r_\zeta} = 0$, $n^{\mathrm{b}}_{j^s_\xi} = s$ and obtain the following analogue of Theorem 6.1:

Theorem 13.1 *In the tropical vertex H, we have a factorization*

$$T_{(1,0),\prod_r \prod_\zeta (1+s_\zeta^r x^r)^r} T_{(0,1),\prod_s \prod_\xi (1+t_\xi^s y^s)^s} = \prod_{b/a \ decreasing} T_{(a,b),F_{a,b}}$$

where

$$F_{a,b} = \sum_{k \geq 0} \Big(\sum_{\substack{|\mathbf{G}_1|=ka, \\ |\mathbf{G}_2|=kb}} \chi(M^{\mathrm{b}}(\mathbf{G}_1, \mathbf{G}_2)) s^{\mathbf{G}_1} t^{\mathbf{G}_2} \Big)(x^a y^b)^k.$$

The commutator formula [3, Theorem 5.6] involves the product

$$T_{(1,0),\prod_r \prod_\zeta (1+s_\zeta^r x^r)} T_{(0,1),\prod_s \prod_\xi (1+t_\xi^s y^s)}$$

on the left hand side (without the additional powers by r and s, respectively) and expresses the factorization on the right hand side in terms of Gromov-Witten invariants $N_{(a,b)}[(\mathbf{G}_1, \mathbf{G}_2)]$ of orbifold blow-ups of the open surface $X_{a,b}^o$. Thus, the corresponding refinement of the GW/Kronecker correspondence is weaker than the versions in the previous sections. Nevertheless, some partial analogues of the localization techniques used in the previous sections are still available (see Section 16.4).

14 Review of localization theory

In the following, let Q be a bipartite quiver with vertices $I \cup J$ and $m(i,j)$ arrows between $j \in J$ and $i \in I$. For a quiver Q we denote by

$$N_q := \{q' \in Q_0 \mid \exists \alpha : q \to q' \vee \alpha : q' \to q\}$$

the set of neighbours of q. Moreover, define $m(i, J) := \sum_{j \in J} m(i, j)$ and $m(I, j)$ analogously.

For a representation X of the quiver Q we denote by $\underline{\dim}X \in \mathbb{N}Q_0$ its dimension vector. Moreover, we choose a level $l : Q_0 \to \mathbb{N}^+$ on the set of vertices. Define two linear forms $\Theta, \kappa \in \mathrm{Hom}(\mathbb{Z}Q_0, \mathbb{Z})$ by $\Theta(d) = \sum_{j \in J} l(j)d_j$ and $\kappa(d) = \sum_{q \in Q_0} l(q)d_q$.

Finally, we define a slope function $\mu : \mathbb{N}Q_0 \to \mathbb{Q}$ by

$$\mu(d) = \frac{\Theta(d)}{\kappa(d)}.$$

For a representation X of the quiver Q we define $\mu(X) := \mu(\underline{\dim}X)$.

Definition 14.1 *A representation X of Q is semistable (resp. stable) if for all proper subrepresentations $0 \neq U \subsetneq X$ the following holds:*

$$\mu(U) \leq \mu(X) \ (resp. \ \mu(U) < \mu(X)).$$

Fixing a slope function as above, we denote by $R_d^{\Theta-\mathrm{sst}}(Q)$ the set of semistable points and by $R_d^{\Theta-\mathrm{st}}(Q)$ the set of stable points in the affine variety $R_d(Q)$ of representations of dimension $d \in \mathbb{N}Q_0$. Moreover, let $M_d^{\Theta-\mathrm{st}}(Q)$ (resp. $M_d^{\Theta-\mathrm{sst}}(Q)$) be the moduli space of stable (resp. semistable) representations. Denote by χ the Euler characteristic in singular cohomology. Note that if $\kappa = \dim$, we obtain the usual definition of stability.

In this setup it is easy to check that the stability condition is equivalent to

$$\sum_{i \in I} l(i) d_i' > \frac{\sum_{i \in I} l(i) d_i}{\sum_{j \in J} l(j) d_j} \sum_{j \in J} l(j) d_j'$$

for all subrepresentations of dimension d'.

Following [17] we say that a general representation of dimension d satisfies some property if there exists a non-empty open subset $U \subseteq R_d(Q)$ that satisfies this property. By $d' \hookrightarrow d$ we denote if a general representation of dimension d has a subrepresentation of dimension d'.

Fix a quiver $\mathcal{Q}$ and two subquivers $\mathcal{Q}^1$ and $\mathcal{Q}^2$ such that $\mathcal{Q}^1 \cup \mathcal{Q}^2 = \mathcal{Q}$ and $\mathcal{Q}^1 \cap \mathcal{Q}^2 = \{q\}$ with $q \in \mathcal{Q}_0$. Then the vertex q is called glueing vertex of $\mathcal{Q}^1$ and $\mathcal{Q}^2$. In the following we denote by $\mathcal{Q} = (\mathcal{Q}^1, \mathcal{Q}^2, q)$ if the quiver $\mathcal{Q}$ is obtained by glueing two quivers $\mathcal{Q}^1$ and $\mathcal{Q}^2$ at the vertex q.

Remark 14.2

- We consider the Dynkin quiver $A_n = (\{q_1, \ldots, q_n\}, \{\alpha_i : q_i \to q_{i+1} \mid i = 1, \ldots, n-1\})$. Let (Q, d) be a tuple consisting of a quiver and a dimension vector and let $j \in Q_0$ be a source of level $l(j)$. We call the tuple $((Q, A_{l(j)}, j = q_{l(j)}), \hat{d})$ a simple extension of (Q, d) at j if $\hat{d}_q = d_q$ for all $q \in Q_0$ and $\hat{d}_q = d_j$ for all $q \in (A_{l(j)})_0$. We proceed analogously for sinks i. We denote the tuple obtained by simple extensions at every vertex by $(Q, Q_0, \hat{d})$.

  Now fix some arbitrary level and let (Q, d) be a tuple as above and consider the simple extension $(Q, Q_0, \hat{d})$. Obviously, every representation of (Q, d) defines a representation of $(Q, Q_0, \hat{d})$ just by defining the corresponding maps to be the identity. On this simple extension we fix the linear form which takes the value 1 at every vertex induced by a source $j \in Q_0$ and j itself and the value 0 at every vertex induced by a sink $i \in Q_0$ and i itself. Now it is easy to verify that a representation of (Q, d) is stable if and only if the corresponding representation is stable with respect to the slope function induced by this linear form. Moreover, we get that a representation of dimension $\hat{d}$ is stable if

$$\sum_{i \in I} \hat{d}_i' > \frac{\sum_{i \in I} \hat{d}_i}{\sum_{j \in J} \hat{d}_j} \sum_{j \in J} \hat{d}_j'$$

  for all subrepresentation of dimension $\hat{d}'$. In particular, the whole machinery that is known for moduli spaces of quivers applies in this situation.

Now we prove the localization theorem in a slightly more general form than [20, Corollary 3.15].

Theorem 14.3 *We have*

$$\chi(M_d^{\Theta-\mathrm{st}}(Q)) = \sum_{\tilde{d}} \chi(M_{\tilde{d}}^{\tilde{\Theta}-\mathrm{st}}(\tilde{Q})),$$

where $\tilde{d}$ ranges over all equivalence classes being compatible with d, and the slope function considered on $\tilde{Q}$ is the one induced by the slope function fixed on Q.

Proof. The only difference from [20, Section 3] is the fact that d is not assumed to be Θ-coprime. Inspection of [20, Section 3] shows that this assumption is only required in the proof of [20, Lemma 3.8]. It remains to strengthen this lemma by proving the following: if $\tilde{V}$ is a stable representation of $\tilde{Q}$, the induced representation V of Q is also stable. Semistability is proved in [20, Lemma 3.8], and the exclusion of proper non-zero subrepresentations of V of the same slope can be proved as in [15, Proposition 4.2].

$\square$

We call a finite subquiver $\mathcal{Q}$ of the universal covering quiver $\tilde{Q}$ of Q localization quiver if there exists a dimension vector $d \in \mathbb{N}\mathcal{Q}_0$ such that $M_d^{\Theta-\mathrm{st}}(\mathcal{Q}) \neq \emptyset$. Moreover, fixing such a dimension vector corresponding to a non-empty moduli space we call the tuple $(\mathcal{Q}, d)$ localization data.

Remark 14.4

- A localization data comes along with a colouring of the arrows $c : \mathcal{Q}_1 \to Q_1$ such that arrows which have the same sink or source are coloured differently. Obviously, every such colouring of the arrows gives rise to a localization data. We call a localization data without a fixed embedding uncoloured. Fixing a dimension vector $d \in \mathbb{N}Q_0$ we denote by $\mathcal{L}_d(Q)$ the set of uncoloured localization data of dimension type d, i.e. $\sum_{q \in \mathcal{Q}_0(i)} \tilde{d}_q = d_i$ for $(\mathcal{Q}, \tilde{d}) \in \mathcal{L}_d(Q)$ where $\mathcal{Q}_0(i)$ denotes the set of vertices corresponding to i. Moreover, for $(\mathcal{Q}, \tilde{d}) \in \mathcal{L}_d(Q)$ we denote by $c_Q(\mathcal{Q})$ the set of colourings. Then Theorem 4.2 can be stated as

$$\chi(M_d^{\Theta-\mathrm{st}}(Q)) = \sum_{(\mathcal{Q}, \tilde{d}) \in \mathcal{L}_d(Q)} |c_Q(\mathcal{Q})| \chi(M_{\tilde{d}}^{\tilde{\Theta}-\mathrm{st}}(\mathcal{Q})).$$

Let $(\mathcal{Q}, d)$ be an uncoloured localization data such that $\mathcal{Q} = (\mathcal{Q}^1, \mathcal{Q}^2, i)$ for some sink i. Let d^1 and d^2 be the corresponding dimension vectors. Let X be a general stable representation of dimension d and X_k, $k = 1, 2$, be the corresponding subrepresentations of $\mathcal{Q}^k$. Then there exists a short exact sequence

$$0 \to X_1 \to X \to \overline{X}_2 \to 0$$

where $\overline{X}_2$ is given by $0 \to S_i^{d_i} \to X_2 \to \overline{X}_2 \to 0$. Let $\underline{\dim}\overline{X}_2 = \bigoplus_{k=1}^l d_k^2$ be the canonical decomposition. Let $q(i) \in Q_0$ be the vertex corresponding to i. We split up $(1, \ldots, l)$ into $n(i) = m(q(i), J) - |N_i| + 1$ (possibly empty) disjoint subsets $S_1, \ldots, S_{n(i)}$ with $\cup S_i = (1, \ldots, l)$. Define $d_{S_t} := \sum_{k \in S_t} d_k^2$ and $\mathcal{Q}_t^2 := \mathrm{supp}(d_{S_t})$. Then we have the following lemma where we refer to [17] for a more detailed discussion of canonical decompositions of dimension vectors:

Lemma 14.5 *The induced tuple*

$$(\hat{\mathcal{Q}}, \hat{d}) := ((\mathcal{Q}^1, \bigcup_{k=1}^{n(i)} \mathcal{Q}_k^2, i), d^1 + \sum_{k=1}^{n(i)} d_{S_k}^2)$$

is an uncoloured localization data.

Proof. Disregarding colourings, we can view $\hat{Q}$ as a subquiver of $\tilde{Q}$ by definition. A general representation of dimension $\underline{\dim}\,\overline{X}_2$ decomposes into representations of dimensions $d_1^2, \ldots, d_l^2$, say $X_{d_1^2}, \ldots, X_{d_l^2}$. Define $X_{S_t} := \oplus_{k \in S_t} X_{d_k^2}$. We can understand the representation $\tilde{X}$ which is given by the short exact sequence

$$0 \to X_1 \to \tilde{X} \to \bigoplus_{k=1}^{n(i)} X_{S_k} \to 0$$

induced by $0 \to X_1 \to X \to \overline{X}_2 \to 0$ as a representation of $(\hat{\mathcal{Q}}, \hat{d})$. Now every subrepresentation of $\tilde{X}$ naturally induces a subrepresentation of X of the same dimension. Thus, since X is stable, $\tilde{X}$ is also stable.

$\square$

Example 14.6

- Consider the generalized Kronecker quiver with dimension vector $(3, 5)$ and the localization data given by
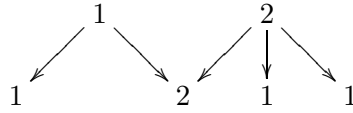
  Now by applying the lemma to the subquiver on the right hand side we obtain the localization data
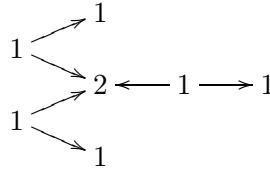
  Note that it is not so obvious how to get from the localization data below to the one above.

Denote by $S(n)$ the n-subspace quiver, i.e. $S(n)_0 = \{i, j_1, \ldots, j_n\}$ and $S(n)_1 = \{\alpha_k : j_k \to i \mid k = 1, \ldots, n\}$.

Lemma 14.7 *Let d be a Schur root of $S(n)$. Then $(S(n), d = (d_0, d_1, \ldots, d_n))$ is an un-coloured localization data of the n-Kronecker quiver with dimension vector $(d_0, \sum_{k=1}^n d_k)$. If we fix a level on the vertices such that $l(i) = \lambda_1$ and $l(j_k) = \lambda_2$ for all $k = 1, \ldots, n$ then $(S(n), d = (d_0, d_1, \ldots, d_n))$ is a localization data concerning to the stability induced by this level.*

Proof. Following [17, Theorem 6.1] for every $d' \hookrightarrow d$ with $d' = (d_0', d_1', \ldots, d_n')$ we have

$$\langle d', d \rangle - \langle d, d' \rangle > 0.$$

But it is straightforward to check that this is equivalent to

$$\frac{\sum_{k=1}^n d_k}{\sum_{k=0}^n d_k} > \frac{\sum_{k=1}^n d_k'}{\sum_{k=0}^n d_k'}$$

which is equivalent to

$$\frac{\lambda_2 \sum_{k=1}^n d_k}{\lambda_2 \sum_{k=1}^n d_k + \lambda_1 d_0} > \frac{\lambda_2 \sum_{k=1}^n d_k'}{\lambda_2 \sum_{k=1}^n d_k' + \lambda_1 d_0'}.$$

$\square$

15 Localization for $K(l_1, l_2)$

We again consider the quiver $K(l_1, l_2)$ where we concentrate on connecting the Euler characteristic of the corresponding moduli spaces to the one of moduli spaces of the Kronecker quiver.

We denote the unique arrow going from j to i by $\alpha_{j,i}$. Without loss of generality we may assume that $l_1 \geq l_2$. Let $(\mathbf{P}_1, \mathbf{P}_2) \in \mathbb{N}K(l_1, l_2)_0$ and (a, b) its Kronecker type, i.e. $|\mathbf{P}_1| = a$ and $|\mathbf{P}_2| = b$, for not necessarily coprime a and b. Define

$$\mathcal{L}_{(a,b)}(K(l_1, l_2)) = \bigcup_{|\mathbf{P_1}|=a, |\mathbf{P_2}|=b} \mathcal{L}_{\mathbf{P_1}, \mathbf{P_2}}(K(l_1, l_2)).$$

For the Kronecker quiver $K(l_1)$ we denote by $\mathcal{L}^{l_2}_{(a,b)}(K(l_1))$ those uncoloured localization quivers $\mathcal{Q}$ such that $|N_i| \leq l_2$ for all sinks $i \in \mathcal{Q}_0$. Note that the stability conditions on $\tilde{K}(l_1, l_2)$ and $\tilde{K}(l_1)$ coincide. Thus, forgetting the colouring (of the vertices), every $\mathcal{Q} \in \mathcal{L}_{(a,b)}(K(l_1, l_2))$ can be understood as an element of $\mathcal{L}^{l_2}_{(a,b)}(K(l_1))$. Moreover, since there only exists at most one arrow between any two vertices, the set of localization data of dimension type $(\mathbf{P_1}, \mathbf{P_2})$ of $K(l_1, l_2)$ equals $\mathcal{L}_{\mathbf{P_1}, \mathbf{P_2}}(K(l_1, l_2))$. Thereby, recall that the vertices of uncoloured localization quivers correspond to vertices of the original quiver. Thus it is straightforward that we have the following lemma:

Lemma 15.1 *There exists a one-to-one correspondence between $\mathcal{L}_{(a,b)}(K(l_1, l_2))$ and tuples $(\mathcal{Q}, c : \mathcal{Q} \to K(l_1, l_2)_1)$ where $\mathcal{Q} \in \mathcal{L}^{l_2}_{(a,b)}(K(l_1))$ and $c : \mathcal{Q}_1 \to K(l_1, l_2)_1$ is a colouring such that arrows which have the same sink or source are coloured differently.*

We get the following statement:

Theorem 15.2 *Fix a Kronecker type (a, b) of $K(l_1, l_2)$. Then we have*

$$\sum_{|\mathbf{P_1}|=a, |\mathbf{P_2}|=b} \chi(\mathbf{P}_1, \mathbf{P}_2) = \sum_{(\mathcal{Q}, \tilde{d}) \in \mathcal{L}^{l_2}_{(a,b)}(K(l_1))} |c_{K(l_1, l_2)}(\mathcal{Q})| \chi(M^s_{\tilde{d}}(\mathcal{Q})).$$

If $l_1 = l_2 =: m$, we have $|c_{K(m,m)}(\mathcal{Q})| = m|c_{K(m)}(\mathcal{Q})|$ for all uncoloured localization data $\mathcal{Q}$. In particular, we have

$$\sum_{|\mathbf{P_1}|=a, |\mathbf{P_2}|=b} \chi(\mathbf{P}_1, \mathbf{P}_2) = m\chi(M^s_{a,b}(K(m))).$$

Proof. The first statement follows from Lemma 15.1 and the considerations from above. Thus assume that $l_1 = l_2$. We choose a map $\mathcal{C} : K(m, m)_1 \to K(m)_1$ such that $\mathcal{C}(\alpha_{j,i}) \neq \mathcal{C}(\alpha_{j',i})$ and $\mathcal{C}(\alpha_{j,i}) \neq \mathcal{C}(\alpha_{j,i'})$ for all vertices $i, i', j, j' \in K(m, m)_0$ with $i \neq i'$ and $j \neq j'$. Every colouring $c : \mathcal{Q}_1 \to K(m, m)_1$ gives rise to a colouring $\mathcal{C} \circ c : \mathcal{Q}_1 \to K(m)_1$. Thus we obtain a surjective map $\mathcal{F} : c_{K(m,m)}(\mathcal{Q}) \to c_{K(m)}(\mathcal{Q})$. We will proceed by induction on the number of sources of some localization quiver in order to show that $|\mathcal{F}^{-1}(c)| = m$ for all colourings $c \in c_{K(m)}(\mathcal{Q})$. If $n = 1$, the statement is straightforward. Let $\mathcal{Q}$ be a subquiver of $\tilde{K}(m)$ (i.e. we have already fixed some colouring) and $\mathcal{Q}_i$, $i = 1, \ldots, m$, be the corresponding subquivers of $\tilde{K}(m, m)$. If we glue a coloured subquiver of type $(j, i_1, \ldots, i_k)$ with $k \leq m$ to $\mathcal{Q}$, the map $\mathcal{C}$ uniquely determines the colours of the extensions of $\mathcal{Q}_i$. Indeed, the glueing vertex corresponds to one of the m sinks of $K(m, m)$ which only depends on the choice of the previous colouring.

$\square$

Note that if $l_2 < l_1$ the induction step fails since the map $\mathcal{F}$ is not surjective in general. Indeed, every sink has only l_2 neighbours so that there might be no arrow $\alpha_{j,i}$ that is coloured as needed.

We consider the quiver $K(l_1, l_2)$ with Kronecker type $(d, d-1)$. Reflecting at every source we may also consider the case $((l_1 - 1)d + 1, d)$ (see also Remark 16.9). By proceeding as in [20, Lemma 6.5], for all localization data $(\mathcal{Q}, d)$ we obtain that $d_q = 1$ for all $q \in \mathcal{Q}_0$. Note that if $l_1 \geq l_2$ this also follows from the considerations from above. If $l_2 > l_1$, the same proof is applicable because it is completely independent of the number of neighbours of some sink. In particular, it only depends on the slope and the number of neighbours of sources.

Theorem 15.3 *We have*

$$\sum_{|\mathbf{P_1}|=d, |\mathbf{P_2}|=d-1} \chi(\mathbf{P_1}, \mathbf{P_2}) = \frac{l_1 l_2}{d((l_1 - 1)d + 1)} \binom{(l_2 - 1)(l_1 - 1)d + l_2 - 1}{d - 1}.$$

Proof. We proceed analogously to [20, Theorem 6.6]. From the considerations from above we obtain that all sub-localization data of a localization data, which have one source, have vertex set $\{j, i_1, \ldots, i_{l_1}\}$ with $d_j = d_{i_k} = 1$. In particular, the moduli spaces of the all considered quivers are points.

There exists exactly one possibility to colour the arrows of such a quiver taking into account the symmetries of S_{l_1}. Now we can glue k subquivers on each vertex i_l, $1 \leq l \leq l_1$, with $0 \leq k \leq (l_2 - 1)$. But we have to take note of the symmetries of S_k. Assuming that there is only one starting vertex to which we can glue, let $y(x)$ be the generating function of such quivers and consider

$$
\begin{aligned}
\phi(x) & = 1 + \frac{(l_2 - 1)}{|S_1|} x^{l_1 - 1} + \frac{(l_2 - 1)(l_2 - 2)}{|S_2|} x^{2(l_1 - 1)} + \ldots + \frac{\prod_{i=1}^{l_2 - 1}(l_2 - i)}{|S_{l_2 - 1}|} x^{(l_2 - 1)(l_1 - 1)} \\
& = \sum_{i=0}^{l_2 - 1} x^{i(l_1 - 1)} \binom{l_2 - 1}{i} = (1 + x^{l_1 - 1})^{l_2 - 1}.
\end{aligned}
$$

The generating function of such trees satisfies the functional equation $y(x) = x(\phi(y(x)))$. The generating function for all localization data is obtained as follows: we start with a localization data of Kronecker type $(1, l_1)$ having l_1 vertices to which we can glue. The resulting generating function is $y(x)^{l_1}$. By applying the Lagrange inversion theorem, see for instance [18] for more details, we obtain that

$$[x^n] y(x)^{l_1} = \frac{l_1}{n} [u^{n - l_1}] \phi(u)^n = \frac{l_1}{n} \binom{n(l_2 - 1)}{\frac{n - l_1}{l_1 - 1}}.$$

If we assign the weight 0 to the source of the localization data started with, every such quiver that has $(l_1 - 1)d + 1$ knots corresponds to a localization data of Kronecker type $(d, (l_1 - 1)d + 1)$. The other way around, we may assume that every localization data has some source $j \in J$ with weight 0 what gives us d choices. This means for every localization data we exactly get d trees. Moreover, we have to take into account that we have l_2 choices for the colour of the source of the starting quiver. Hence we get

$$\sum_{|\mathbf{P_1}|=(l_1-1)d+1, |\mathbf{P_2}|=d} \chi(\mathbf{P_1}, \mathbf{P_2}) = \frac{l_1 l_2}{d((l_1 - 1)d + 1)} \binom{(l_2 - 1)(l_1 - 1)d + l_2 - 1}{d - 1}.$$

$\square$

16 Vanishing of the Euler characteristic

Assume that

$$\sum_{i \in I} l(i) d_i = K \sum_{j \in J} l(j) d_j$$

for some $K \in \mathbb{Q}$. Let $S(n)^t$ be the quiver obtained from the subspace quiver by reversing all arrows.

Definition 16.1 *Let $d \in \mathbb{N}Q_0$ and $\mathrm{supp}(d)_0 = \{i_1, \ldots, i_n, j_1, \ldots, j_m\}$. We say that d satisfies the no-peak condition if the following conditions hold: If $m = 1$, then there exists no decomposition of the form*

$$d = (d_j, d_{i_1}^1 + \ldots + d_{i_1}^{m(i_1,j)}, \ldots, d_{i_n}^1 + \ldots + d_{i_n}^{m(i_n,j)})$$

such that $d = (d_j, d_{i_1}^1, \ldots, d_{i_1}^{m(i_1,j)}, \ldots, d_{i_n}^1, \ldots, d_{i_n}^{m(i_n,j)})$ is a Schur root of $S(m(I,j))$. If $n = 1$, then there exists no decomposition of the form

$$d = (d_i, d_{j_1}^1 + \ldots + d_{j_1}^{m(i,j_1)}, \ldots, d_{j_m}^1 + \ldots + d_{j_m}^{m(i,j_m)})$$

such that $d = (d_i, d_{j_1}^1, \ldots, d_{j_1}^{m(i,j_1)}, \ldots, d_{j_m}^1, \ldots, d_{j_m}^{m(i,j_m)})$ is a Schur root of $S(m(i,J))^t$.

In other words, if d satisfies the no-peak condition, there does not exist a localization data with only one sink or only one source. For instance if $Q = K(m)$ and $d = (a, a)$, then d satisfies the no-peak condition for all $a \geq 2$.

Proposition 16.2 *For every sink i of some localization quiver with $|I| \geq 2$ we have*

$$l(i) < K \sum_{j \in N_i} l(j).$$

For every source j of some localization quiver with $|J| \geq 2$ we have
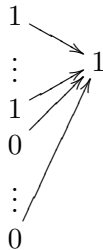
$$l(j) < \frac{1}{K} \sum_{i \in N_j} l(i).$$

Proof. Assume that there exists a subquiver

$$
\begin{array}{c}
j_1 \\
j_2 \searrow \\
\vdots \quad \rightarrow i \\
j_n \nearrow
\end{array}
$$
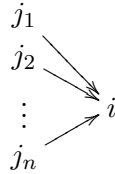
By use of

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}(X, Y)$$

for two representations of a quiver Q, see for instance [16], we get that a representation of this quiver has a factor of type

$$
\begin{array}{c}
1 \\
\vdots \searrow \\
1 \quad 1 \\
0 \nearrow \\
\vdots \\
0
\end{array}
$$

if

$$\sum_{k=1}^{s} d_{j_k} + d_i - \sum_{k=1}^{n} d_{j_k} = -\sum_{k=s+1}^{n} d_{j_k} + d_i > 0. \tag{2}$$

Here s denotes the number of vertices of dimension one on the left hand side of the quiver. Note that we may without lose of generality assume that every homomorphism is surjective because otherwise we would get a surjection for some $t < s$.

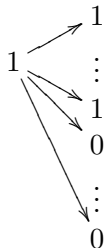Since such a factor has to be of bigger slope we get

$$l(i) < K \sum_{k=1}^{s} l(j_k).$$

Since inequality (2) definitely holds for $s = n$, the claim follows.

The second claim follows when considering subquivers of type
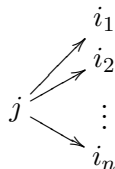
and subrepresentations of type

Since we now deal with subrepresentations which must be of smaller slope we obtain

$$Kl(j) < \sum_{k=1}^{s} l(i_k).$$

$\square$

We obtain the following result:

Theorem 16.3 *Assume that $l(i) = \lambda_1$ for all $i \in I$, $l(j) = \lambda_2$ for all $j \in J$ and let $K = \frac{\lambda_1}{\lambda_2}$. Moreover, assume that*

$$\sum_{i \in I} l(i)d_i = K \sum_{j \in J} l(j)d_j$$

and that d satisfies the no-peak condition. Then there does not exist any localization quiver.

Proof. According to Proposition 16.2 for every vertex of a localization quiver we have

$$l(i) = \lambda_1 < \frac{\lambda_1}{\lambda_2} \sum_{j \in N_i} l(j) = \lambda_1 |N_i|$$

and

$$l(j) = \lambda_2 < \frac{\lambda_2}{\lambda_1} \sum_{i \in N_j} l(i) = \lambda_2 |N_j|.$$

Thus every vertex of a localization quiver is forced to have at least two neighbours. In particular, every localization quiver is forced to be cyclic.

$\square$

If $l(q) = 1$ for all $q \in Q_0$ and $K = 1$, this proves a more general version of [20, Corollary 6.3]:

Corollary 16.4 *Let $d \in \mathbb{N}Q_0$ be a dimension vector such that*

$$\sum_{i \in I} d_i = \sum_{j \in J} d_j,$$

$l(q) = 1$ *for all $q \in Q_0$ and $\sum_{j \in J} d_j \neq 1$. Then we have $\chi(M_d^{\Theta-\mathrm{st}}(Q)) = 0$.*

Moreover, we get the following result:

Theorem 16.5 *Let $d \in \mathbb{N}Q_0$. Assume that*

$$\sum_{i \in I} l(i)d_i = K \sum_{j \in J} l(j)d_j$$

and that d satisfies the no-peak condition. Then there exists no localization quiver if

$$l(i) \geq K \sum_{j \in N_i} m(i,j)l(j)$$

for some sink $i \in I$ or

$$l(j) \geq \frac{1}{K} \sum_{i \in N_j} m(i,j)l(i)$$

for some source $j \in N_i$. In particular, the Euler characteristic of the corresponding moduli space vanishes.

Proof. The claim follows because every neighbour of some vertex of the original quiver gives rise to $m(i,j)$ neighbours in the universal cover.

$\square$

If Q has the generalized Kronecker quiver $K(m)$ as a proper subquiver such that the corresponding sink i has only one neighbour, we immediately get the following corollary:

Corollary 16.6 *Let $d \in \mathbb{N}Q_0$ satisfy the no-peak condition. Assume that $K = 1$ and that there exist $i, j \in Q_0$ such that $m(i,j) = m$ and $N_i = \{j\}$. If $l(j) = 1$ and $l(i) = m$, there does not exist any localization quivers.*

The last case we treat is the following:

Theorem 16.7 *Let $d \in \mathbb{N}Q_0$ satisfy the no-peak condition and assume that $l(i) = l$ for all $i \in I$ and*

$$l \sum_{i \in I} d_i = K \sum_{j \in J} l(j) d_j$$

where $K \in \mathbb{N}$. Moreover, for all $j \in J$ and $i \in I$ let

$$Kl(j) \geq l\left(m(I, j) - 1\right).$$

Then there exists no localization quiver.

Proof. Because of Theorem 16.5 we can assume that

$$Kl(j) = l\left(m(I, j) - 1\right)$$

for all vertices $i \in I$ and $j \in J$.

Assume that there exists a localization quiver and let X be a stable representation. Consider a subrepresentation

such that $|N(i_k)| = 1$ for all $k \geq 2$. Then we obviously have $d_{i_k} \leq d_j$ for all $k \geq 2$. Consider the kernel of the map X_k. We have

$$
\begin{aligned}
\dim(\sum_{t=1}^{N} X_t(\ker(X_k))) \quad &\leq \quad \dim(\ker(X_k))(N-1)l \leq \\
&\leq \quad \dim(\ker(X_k))Kl(j).
\end{aligned}
$$

Thus we obtain $\ker(X_k) = \{0\}$ and, therefore, $d_j = d_{i_k}$ for all $k = 2, \ldots, N$. Thus by Lemma 14.5 we may assume that $d_j = d_{i_k} = 1$.

But we also have that $N = m(I, j)$. Indeed, otherwise we would have

$$\dim(\sum_{l=1}^{N} X_l(X_j)) = Nl \leq l\left(m(I, j) - 1\right) = Kl(j).$$

But this means that the representation has a factor representation of dimension type $(l(j), Kl(j))$ which contradicts the stability condition.

$\square$

Moreover, we get the following corollary:

Corollary 16.8 *Assume that $l(q) = 1$ for all vertices of the quiver Q and that*

$$K \sum_{j \in J} d_j = \sum_{i \in I} d_i$$

where $\sum_{j \in J} d_j \neq 1$ and $K \in \mathbb{N}$. If we have $m(I, j) \leq K + 1$ for all $j \in J$, there exists no localization quiver. In particular, the Euler characteristic of the corresponding moduli space vanishes.

Remark 16.9

- As far as the Kronecker quiver $K(m)$ is concerned the preceding statement says that the Euler characteristic vanishes if $(d, e) = (d, kd)$ and $k \geq m - 1$ which also follows from Theorem 16.4 by applying the reflection functor, see [1] for the definition. In the case of bipartite quivers such that $l(q) = 1$ for all q, the reflection functor applied to all sinks or sources simultaneously gives rise to isomorphisms between moduli spaces. Indeed, it is checked easily that the stability conditions are equivalent. Moreover, subrepresentations become factor representations and vice versa.

  But in the general case, it is not obvious how to get isomorphisms between moduli spaces corresponding to different dimension vectors (except the one coming from transposing all maps) because the stability conditions are not compatible.

17 Further examples

In this section we give several examples and applications illustrating the results of the preceding sections.

17.1 The case K=1

Assume that we have

$$\sum_{i \in I} l(i) d_i = \sum_{j \in J} l(j) d_j.$$

Let $(\mathcal{Q}, d)$ be a localization data and let
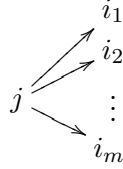
be a subquiver such that $N_{i_k} = \{j\}$ for all $k = 2, \ldots, m$. Because of the stability condition we have

$$\sum_{k=2}^{m} l(i_k) d_{i_k} < l(j) d_j.$$

In particular, if $l(j) = 1$, by Lemma 14.5 it follows that $d_{i_k} = 0$ for all $k = 2, \ldots, m$. Indeed, in the canonical decomposition of $(d_j, d_{i_2}, \ldots, d_{i_k})$ the simple representation S_j at least occurs with multiplicity $r := d_j - \sum_{k=2}^{m} l(i_k) d_{i_k}$. In particular, we also get a factor of slope $(d_j - r, \sum_{k=2}^{m} l(i_k) d_{i_k}) = (d_j - r, d_j - r)$ which contradicts the stability condition.

Recall also Proposition 16.2 which for $K = 1$ says how many neighbours of which kind are allowed for a vertex with a fixed level.

Now assume that there exists a subquiver
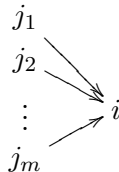
such that $l(i) = 1 = l(j_k)$ for all k and $N_i = \{j_1, \ldots, j_m\}$. Then we have

$$(\sum_{k=1}^{m} d_{j_k}) - d_{j_l} \geq d_i$$

for all $l = 1, \ldots, m$ because otherwise a representation of this dimension would have a factor isomorphic to $1 \to 1$. Moreover, it again follows that we have $|N_{j_k}| \geq 2$ for all k.

In the following we assume that every vertex has enough neighbours in order to show that in general it can not be assumed that there exist no localization data.

We restrict to the cases $l(j) = 1$ for all $j \in J$ and $l(i) \neq 1$ for all $i \in I$ and assume that $\sum_{j \in J} d_j = \sum_{i \in I} l(i) d_i$ with $d_i \geq 2$ for all $i \in I$. First we deal with the Kronecker quiver with dimension vector (e, d) and $l(i)e = d$. Then the stability condition is

$$l(i)e' > \frac{l(i)e}{d} d' \Leftrightarrow e' > \frac{e}{d} d'$$

for all $(d', e') \hookrightarrow (d, e)$. Thus we can consider the localization quiver
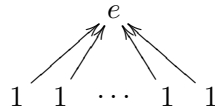
with d vertices on the bottom row. Since we have $d > e$, it is easy to check that this is a Schur root of the subspace quiver. Note that if $l(i) = 1$ there only exist semistable and polystable points respectively. But nevertheless, there can exist localization quivers in the case of bipartite quivers with more than one sink and such that $l(i) = 1$ not for all sinks of the quiver, see also Example 17.1.

Now consider two dimension vectors (d_i, e_i) satisfying the conditions from above. Then we can glue the mentioned localization quivers in order to get one of type $(d_1 + d_2, e_1 + e_2)$ as follows
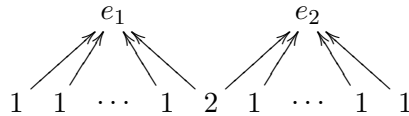
Now it is easy to check that a general representation of this quiver and with this dimension vector is stable. Moreover, we can recursively construct such quivers. Thus we get localization data for all quivers as above such that $\sum_{j \in J} d_j = \sum_{i \in I} l(i) d_i$ and $l(i) \neq 1$ for all $i \in I$.

Example 17.1

Consider the quiver having vertices $Q_0 = \{j, i_1, i_2\}$ with $l(j) = 1 = l(i_1)$ and $l(i_2) = l$. Moreover, consider the dimension vector $(4l + 2, 2, 4)$ and the localization data
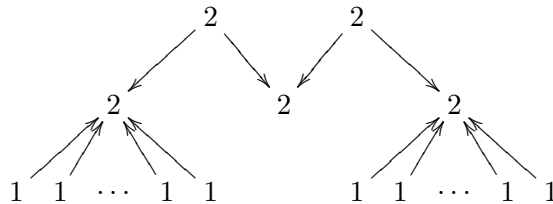
with twice $2l - 1$ vertices on the bottom row and where the single vertex in the middle row has level one. Now it is easily verified that a general representation of this dimension is stable.

Example 17.2

The vanishing of the Euler characteristic does not imply that the moduli space of the original quiver is empty. For instance let Q be the quiver with vertices $J = \{j\}$, $I = \{i_1, i_2\}$ and $m(i_1, j) = 3$ and $m(i_2, j) \geq 3$. Consider the dimension vector d given by $d_j = 21$, $d_{i_1} = 6$ and $d_{i_2} = 3$ and the level given by $l(j) = 1$, $l(i_1) = 3$ and $l(i_2) = 1$. Thus we have $l(i_1) = m(i_1, j)l(j)$ and by Theorem 16.5 it follows that $\chi(M_d^{\Theta-\mathrm{st}}(Q)) = 0$. But a general representation of the following subquiver of the universal abelian (!) covering quiver
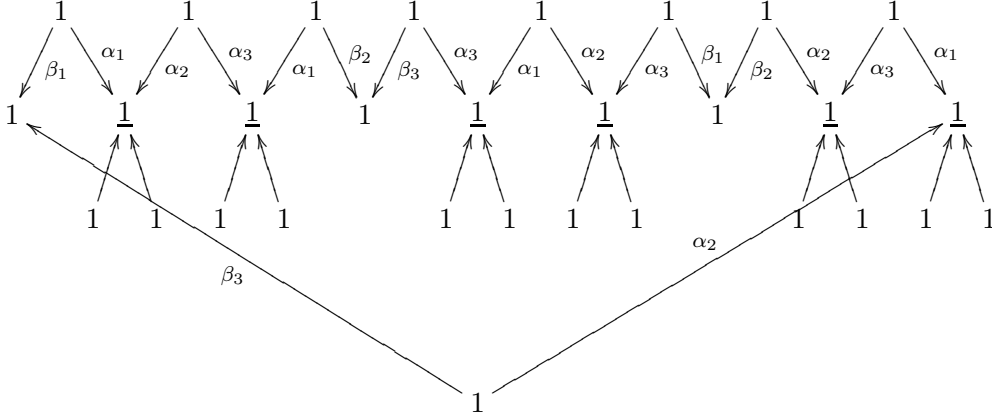
is stable. Here we can choose every appropriate colouring such that different vertices have different weights. Moreover, we denote the arrows between j and i_1 by α_k, the arrows between j and i_2 by β_k and by _ we denote the vertices corresponding to i_1. Now it is easy to verify that a general representation of this localization data (corresponding to the universal abelian cover) is stable.

Note that the number of arrows $m(j, i_1)$ plays an important role. If we choose $m(j, i_1) = 5$ in the same setup, there exists for instance the following localization data
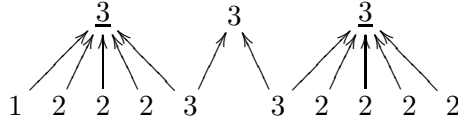
Note that, the dimension vector $(3, 1, 2, 2, 2, 2, 2, 2, 2)$ is a Schur root of the 8-subspace quiver. Thus, it is straightforward to check that a general representation of this dimension is stable.

17.2 The case $Q = K(l_1, l_2)$

We consider the case $m(i, j) = 1$ and $l(q) = 1$ for all $i, j, q \in Q_0$. Let $d \in \mathbb{N}Q_0$ and define $|J| = l_2$, $|I| = l_1$ and $b = \sum_{j \in J} d_j$, $a = \sum_{i \in I} d_i$. We may assume that $l_1 \geq l_2$. Otherwise we turn around all arrows. Moreover, we can assume that $\frac{l_1}{2} b \geq a$. Otherwise we apply the reflection functor to every sink and turn around all arrows afterwards. Note that, for the reflected Kronecker type $(b, l_1 b - a)$ we have

$$\frac{l_1 b - a}{b} < \frac{a}{b} \Leftrightarrow \frac{l_1}{2} b < a.$$

Let d be a root of Q and let $m := \lceil \frac{l_2 + l_1}{2} \rceil$. By an easy calculation we get that $f(d_1, \ldots, d_{l_2}) := \sum_{j \in J} d_j^2$ has a unique minimum at $d_j = \frac{b}{l_2}$, $j \in J$. Thus we get

$$
\begin{aligned}
\langle d, d \rangle &= \sum_{j \in J} d_j^2 + \sum_{i \in I} d_i^2 - ab \geq \frac{b^2}{l_2} + \frac{a^2}{l_1} - ab = \frac{1}{l_1 l_2}(l_1 b^2 + l_2 a^2 - l_2 l_1 ab) \\
&= \frac{1}{l_2}\left(b^2 + \frac{l_2}{l_1}a^2 - l_2 ab\right) \geq \frac{2}{l_2 + l_1}\left(b^2 + a^2 - \frac{l_2 + l_1}{2}ab\right)
\end{aligned}
$$

where the last inequality holds because $l_1 \geq l_2$ and

$$
\frac{l_2}{l_1}a^2 - l_2 ab \geq a^2 - \frac{l_2 + l_1}{2}ab \Leftrightarrow \frac{l_1}{2}b \geq a.
$$

If $\langle d, d \rangle \leq 0$, it obviously follows that (b, a) is a root of $K(m)$.

Now we consider the case $l_1 = l_2 =: m$ and $\langle d, d \rangle = 1$ in greater detail. We restrict to the case $m \neq 1$. Then we have $m \nmid b$ or $m \nmid a$ because otherwise we would have

$$
(k_1^2 m^3 + k_2^2 m^3 - m^4 k_1 k_2) = m^3(k_1^2 + k_2^2 - m k_1 k_2) = m^2 \langle d, d \rangle
$$

for some $k_1, k_2 \in \mathbb{N}$. Thus we get that the first inequality is proper and we may assume that $m \nmid b$. Then we have $d_j = \frac{b + s_j}{m}$ with $s_j \neq 0$ and $\sum_{j=1}^m s_j = 0$ because $\sum_{j=1}^m d_j = \sum_{j=1}^m d_j + s_j = b$. We obtain

$$
\sum_{j=1}^m d_j^2 = \sum_{j=1}^m \left(\frac{b + s_j}{m}\right)^2 = \frac{b^2}{m} + \sum_{j=1}^m \frac{2bs_j + s_j^2}{m^2} = \frac{b^2}{m} + \sum_{j=1}^m \frac{s_j^2}{m^2}.
$$

Now we can assume that $s_j > 0$ for $j = 1, \ldots, n$ and $s_j < 0$ for $j = n+1, \ldots, m$ with $n \geq 1$. Define n by $\frac{n}{m} = \lceil \frac{b}{m} \rceil - \frac{b}{m}$. Then we have $\frac{m-n}{m} = \frac{b}{m} - \lfloor \frac{b}{m} \rfloor$. If we choose $s_j = n - m$ for all $j = 1, \ldots, n$ and $s_j = n$ for all $j = n+1, \ldots, m$, we get the unique tuple satisfying $\sum_{j=1}^m s_j = 0$ and $|s_j| \leq m$ for every j. Because

$$
(s_k - m)^2 + (s_l + m)^2 = s_k^2 + s_l^2 - 2s_k m + 2s_l m + 2m^2
$$

and

$$
s_l m - s_k m + m^2 \geq 0 \Leftrightarrow s_k + m \geq s_l,
$$

it follows that $\sum_{j=1}^m s_j^2$ takes its minimum at this tuple.

Keeping in mind that $\sum_{j=1}^n s_j = -\sum_{j=n+1}^m s_i$ as before we get

$$
\begin{aligned}
\sum_{j=1}^n s_j^2 + \sum_{j=n+1}^m s_j^2 &\geq n\left(\frac{\sum_{j=1}^n s_j}{n}\right)^2 + (m-n)\left(\frac{\sum_{j=n+1}^m s_j}{m-n}\right)^2 \\
&= \frac{m\left(\sum_{j=1}^n s_j\right)\left(-\sum_{j=n+1}^m s_j\right)}{n(m-n)} \geq \frac{mn^2(m-n)^2}{n(m-n)} \geq m(m-1)
\end{aligned}
$$

where the last inequality holds because $m \geq n+1$. Thus we get

$$
1 = \langle d, d \rangle = \frac{b^2}{m} + \sum_{i=1}^m \frac{2bs_j + s_j^2}{m^2} + \frac{a^2}{m} + \sum_{j=1}^m \frac{2at_j + t_j^2}{m^2} - ab \geq \frac{m(m-1)}{m^2} + \frac{\langle (b,a), (b,a) \rangle}{m}
$$

and, therefore,

$$
\langle (b, a), (b, a) \rangle \leq m - (m-1) = 1.
$$

Thus (b, a) is a root of $K(m)$. We get the following statement:

Lemma 17.3 *If* $(\mathbf{P_1}, \mathbf{P_2})$ *is a root of Kronecker type* (b, a), *then* (b, a) *is a root of* $K(m)$. *If* $M^{\Theta-\mathrm{st}}_{(b,a)}(K(m))^T \neq \emptyset$, *then there exists a dimension vector* $(\mathbf{P'_1}, \mathbf{P'_2})$ *of Kronecker type* (b, a) *such that* $M^{\Theta-\mathrm{st}}_{(\mathbf{P'_1}, \mathbf{P'_2})}(K(m, m)) \neq \emptyset$.

Proof. The first statement follows from the considerations from above. Thus let (b, a) be a root. By [20, Theorem 3.11] torus fixed point sets are described as moduli spaces of the universal abelian covering quiver. Now applying methods similar to the one of Section 15 we can construct stable representations of $K(l_1, l_2)$ as claimed.

□

Note that even for the universal abelian cover there do not exist stable representations for every dimension vector of $K(m)$, e.g. if $m = 3$ and $(d, e) = (2, 2)$ or $(d, e) = (4, 4)$.

Example 17.4

Consider the example $(3, 5)$ of $K(3)$ with the following localization data:
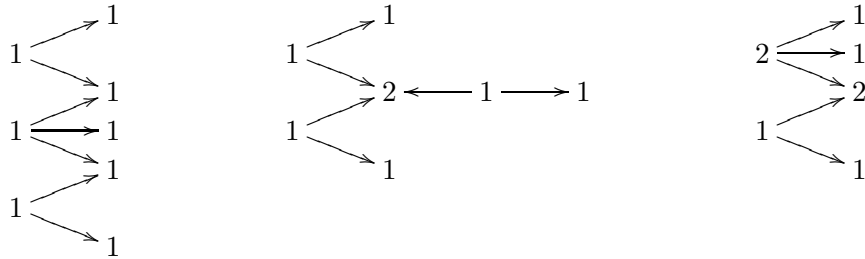
In summary, we have 68 possible colourings of the arrows (i.e. possibilities to embed these quivers into $\tilde{K}(3)$) and, therefore, we get $\chi(M^s_{(3,5)}(K(3))) = 68$.

Now consider $K(3, 3)$ and $\mathbf{P_1} = 3 + 1 + 1$ and $\mathbf{P_2} = 1 + 1 + 1$. In order to obtain some localization data we can think of colouring the vertices of the above quivers such that vertices having the same sink or source do not have the same colour. Obviously, this is just possible for the first localization quiver where we have 12 possibilities (six on the left hand side and two on the right hand side). If we mod out the action of S_2 we get $\chi(3 + 1 + 1, 1 + 1 + 1) = 6$.

Now consider $\mathbf{P_1} = 2 + 2 + 1$ and $\mathbf{P_2} = 1 + 1 + 1$. It is straightforward that we have 18 possibilities to colour the first quiver. But now the second quiver gives also rise to some localization data. Dividing out the action of S_3 we have 6 possible colourings. Thus, in summary, we get $\chi(2 + 2 + 1, 1 + 1 + 1) = 24$. Note that for the last localization quiver there exists no possible colouring.

Moreover, we have $\chi(3 + 2 + 0, 1 + 1 + 1) = 1$ when colouring the localization quiver in the middle, $\chi(3 + 1 + 1, 2 + 1 + 0) = 1$ when colouring the quiver on the left hand side, $\chi(2 + 2 + 1, 2 + 1 + 0) = 3 + 2 = 5$ when colouring the quivers on the left and right hand side. Taking into account the possibilities to obtain ordered partitions induced by the partitions, it is easy to verify $\sum_{|\mathbf{P_1}|=3, |\mathbf{P_2}|=5} \chi(\mathbf{P_1}, \mathbf{P_2}) = 204$ as predicted by Theorem 15.2.

References

[1] Bernstein, J., Gelfand, I.M., Ponomarev, V.A.: Coxeter functors and Gabriel's theorem. Russian Math. Surveys **28**, 17-32 (1973).

[2] Gross, M., Pandharipande, R.: Quivers, curves, and the tropical vertex. Portugalia Math. **67**, 211-259 (2010).

[3] Gross, M., Pandharipande, R., Siebert, B.: The tropical vertex. Duke Math. J. **153**, No. 2, 297-362 (2010).

[4] Engel, J., Reineke, M.: Smooth models of quiver moduli. Math. Z. **262**, 4, 817-848 (2009).

[5] Kac, V.G.: Infinite root systems, representations of graphs and invariant theory. Inventiones mathematicae **56**, 57-92 (1980).

[6] Kontsevich, M., Soibelman, Y.: Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. Preprint 2008. arXiv:0811.2435.

[7] Kontsevich, M., Soibelman, Y.: Affine structures and non-Archimedean analytic spaces. In: The unity of mathematics (P. Etingof, V. Retakh, I.M. Singer, eds.), 321-385, Progr. Math. **244**, Birkhäuser 2006.

[8] Kuniba, A., Nakanishi, T., Tsuboi, Z.: The canonical solutions of the Q-systems and the Kirillov-Reshetikhin conjecture. Comm. Math. Phys. **227**, 155-190 (2002).

[9] Reineke, M.: The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli. Invent Math. **152**, 349-368 (2003).

[10] Reineke, M.: Cohomology of noncommutative Hilbert schemes. Algebr. Represent. Theory. **8**, 541-561 (2005).

[11] Reineke, M.: Poisson automorphisms and quiver moduli. J. Inst. Math. Jussieu **9**, 3, 653-667 (2010).

[12] Reineke, M.: Cohomology of quiver moduli, functional equations, and integrality of Donaldson-Thomas type invariants. Preprint 2009, to appear in Comp. Math.. arXiv:0903.0261

[13] Reineke, M.: Moduli of representations of quivers. In: Trends in Representation Theory of Algebras and Related Topics (ed. A. Skowronski), EMS Series of Congress Reports, EMS Publishing House, 2008.

[14] Reineke, M.: Degenerate Cohomological Hall algebra and quantized Donaldson-Thomas invariants of m-loop quivers. Preprint 2011. arXiv:1102.3978

[15] Reineke, M.: Localization in quiver moduli. J. Reine Angew. Math. **631**, 59-83 (2009).

[16] Ringel, C.M.: Representations of K-species and bimodules. Journal of Algebra **41**, 269-302 (1976).

[17] Schofield, A.: General representations of quivers. Proc. London Math. Soc. (3) **65**, 46-64 (1992).

[18] Stanley, R.P.: Enumerative Combinatorics vol. 2. Cambridge University Press, New York/Cambridge, 1999.

[19] Stoppa, J.: Universal covers and the GW/Kronecker correspondence. Preprint 2010. arXiv:1011.4897

[20] Weist, T.: Localization in quiver moduli spaces. Preprint 2009. arXiv: 0903.5442.